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SPECTRAL PROPERTIES OF CLASS A OPERATORS

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ABSTRACT. Let T be a class A operator. It is shown that (i) every eigenvector of T corresponding to a nonzero eigenvalue is a normal eigenvector, (ii) every approximate eigenvector of T correseponding to a nonzero approximate eigenvalue is a normal approximate eigenvector, and (iii) T is normal if the planar Lebesque measure of the spectrum of T is zero.

1. INTRODUCTION

A bounded linear operator T on a Hilbert space H with inner product $(\cdot,\cdot)$ is said to be p-hyponormal [**Al**], $p > 0$, if $(T^*T)^p \geq (TT^*)^p$. A p-hyponormal operator is said to be hyponormal if $p = 1$, semi-hyponormal if $p = 1/2$. The Löwner-Heinz inequality implies that if T is p-hyponormal, then it is q-hyponormal for any $0 < q \leq p$. Let $T = U|T|$ be the polar decomposition of the operator T and let $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$. The operator $\widetilde{T}$ is known as the Aluthge transform of T. An operator T is said to be w-hyponormal [**AlW1**], [**AlW2**] if

$$|\widetilde{T}| \geq |T| \geq |\widetilde{T}^*|. \tag{1.1}$$

An operator T is said to be a class A operator [**FIY**] if

$$|T^2| \geq |T|^2.$$

Among these classes of operators, the following inclusions are known to hold:

$$\{\text{hyponormal}\} \subset \{p\text{-hyponormal}, p < 1\} \subset \{w\text{-hyponormal}\} \subset \{\text{class A}\}.$$

All these inclusions are known to be proper.

If T is w-hyponormal, it follows from (1.1) that $\widetilde{T}$ is semi-hyponormal. Moreover, if T is semi-hyponormal, then $\widetilde{T}$ is hyponormal [**Al**]. Because of these, one can derive many spectral properties for the class of w-hyponormal operators from those of the hyponormal operators. In this paper, we study certain spectral properties for class A operators. In particular, we obtain several spectral properties that class A operators inherit from those of the w-hyponormal operators. The connection we use to obtain our results is based on a theorem, due to Ito and Yamazaki [**IY**], that if T is class A, then T^2 is w-hyponormal.

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2. Eigenvectors and approximate eigenvectors

Let T be a bounded linear operator. A complex number λ is said to be in the *point spectrum* $\sigma_p(T)$ of T if there is a nonzero vector x for which $(T-\lambda)x = 0$. If in addition $(T^* - \overline{\lambda})x = 0$, then λ is said to be in the *normal point spectrum* $\sigma_{np}(T)$ of T. Thus, $\lambda \in \sigma_{np}(T)$ if there is an eigenvector x corresponding to λ which is a normal eigenvector. A complex number λ is said to be in the *approximate point spectrum* $\sigma_a(T)$ of T if there is a sequence $\{x_n\}$ of unit vectors for which $(T-\lambda)x_n \to 0$. If in addition $(T^* - \overline{\lambda})x_n \to 0$, then λ is said to be in the *normal approximate point spectrum* $\sigma_{na}(T)$ of T. Thus, $\lambda \in \sigma_{na}(T)$ if there is an approximate eigenvector $\{x_n\}$ corresponding to λ which is a normal approximate eigenvector. In general, one has $\sigma_{np}(T) \subset \sigma_p(T)$ and $\sigma_{na}(T) \subset \sigma_a(T)$.

The following facts are known for each class of operators :

(a) Let T be hyponormal or semi-hyponormal.[**Xi**]
- (i) $\sigma_{np}(T) = \sigma_p(T)$ and $\sigma_{na}(T) = \sigma_a(T)$.
- (ii) Let T be hyponormal.Then $T - \lambda$ is hyponormal for any scalar λ, and $\|(T^* - \overline{\lambda})x\| \leq \|(T-\lambda)x\|$
- (iii) Let T be semi-hyponormal. If $(T-\lambda)x = 0$, then $(T^* - \overline{\lambda})x = 0$. and if $(T-\lambda)x_n \to 0$ for $\{x_n\}$ is a sequence of unit vectors, then $(T^* - \overline{\lambda})x_n \to 0$.

(b) Let T be p-hyponormal $(0 < p < \frac{1}{2})$.[**ChH**]
- (i) $\sigma_{np}(T) = \sigma_p(T)$ and $\sigma_{na}(T) = \sigma_a(T)$.
- (ii) If $(T-\lambda)x = 0$, then $(T^* - \overline{\lambda})x = 0$.

(c) Let T be w-hyponormal.[**AlW1**]
- (i) $\sigma_{np}(T)\backslash\{0\} = \sigma_p(T)\backslash\{0\}$. and $\sigma_{na}(T)\backslash\{0\} = \sigma_a(T)\backslash\{0\}$.
- (ii) If $(T-\lambda)x = 0$ and $\lambda \neq 0$, then $(T^* - \overline{\lambda})x = 0$.

(d) Let T be class A.[**Uch**]
- (i) $\sigma_{np}(T)\backslash\{0\} = \sigma_p(T)\backslash\{0\}$.
- (ii) If $(T-\lambda)x = 0$ and $\lambda \neq 0$, then $(T^* - \overline{\lambda})x = 0$.

As note above, every eigenvector of a hyponormal operator or a p-hyponormal $(0 < p < 1)$ operator T is a normal eigenvector. As for w-hyponormal operators or class A operators, we can know that every eigenvector corresponding to a nonzero eigenvalue is a normal eigenvector.

Although every approximate eigenvector $\{x_n\}$ corresponding to an approximate eigenvalue λ of a hyponormal operator or a semi-hyponormal operator T is a normal approximate eigenvector, it is not known if this assertion holds if T is p-hyponormal $(0 < p < \frac{1}{2})$. In the remainder of this section, we show that the assertion does hold for p-hyponormal operators by first proving that the assertion holds for class A operators provided $\lambda \neq 0$.

Theorem 1 ([**IY**, Corollary 5])**.** *If T is class A, then T^2 is w-hyponormal.*

For a bounded linear operator S and scalar μ, the following equations hold:

$$(|S| + |\mu|)(|S| - |\mu|) = S^*(S - \mu) + \mu(S^* - \overline{\mu}), \tag{2.1}$$

$$(|S^*| + |\mu|)(|S^*| - |\mu|) = S(S^* - \overline{\mu}) + \overline{\mu}(S - \mu). \tag{2.2}$$

Note that both the operators $|S| + |\mu|$ and $|S^*| + |\mu|$ are invertible if $\mu \neq 0$.

Theorem 2. *Let T be w-hyponormal, $\lambda \neq 0$ and $\{x_n\}$ be a sequence of unit vectors. If $(T - \lambda)x_n \to 0$, then $(T^* - \overline{\lambda})x_n \to 0$.*

Proof. Since $\|(T - \lambda)x_n\| \geq |\, \|Tx_n\| - |\lambda|\, |$, the assumption implies $\|Tx_n\| \to |\lambda|$. And the semi-hyponormal operator $\widetilde{T}$ satisfies $(\widetilde{T} - \lambda)|T|^{1/2}x_n \to 0$. Since

$$\|T\| \geq \|\, |T|^{1/2}\| \, \|\, |T|^{1/2}x_n\| \geq \|Tx_n\| \to |\lambda|,$$

deleting finitely many terms if necessary, we may assume the sequence $\{|T|^{1/2}x_n\}$ is both bounded and bounded below:

$$M \geq \|\, |T|^{1/2}x_n\| \geq m > 0$$

for all n. Set $y_n = |T|^{1/2}x_n / \|\, |T|^{1/2}x_n\|$. Then $\{y_n\}$ is a sequence of unit vectors and

$$\begin{aligned} \|(\widetilde{T} - \lambda)y_n\| &= \|\, |T|^{1/2}x_n\|^{-1} \, \|(\widetilde{T} - \lambda)|T|^{1/2}x_n\| \\ &\leq m^{-1}\|(\widetilde{T} - \lambda)|T|^{1/2}x_n\| \to 0. \end{aligned}$$

It follows from (a)-(ii) that $(\widetilde{T}^* - \overline{\lambda})y_n \to 0$. Therefore,

$$\begin{aligned} \|(\widetilde{T}^* - \overline{\lambda})|T|^{1/2}x_n\| &= \|\, |T|^{1/2}x_n\| \|(\widetilde{T}^* - \overline{\lambda})y_n\| \\ &\leq M \, \|(\widetilde{T}^* - \overline{\lambda})y_n\| \to 0 \end{aligned}$$

and hence $(\widetilde{T}^* - \overline{\lambda})|T|^{1/2}x_n \to 0$.

Setting $S = \widetilde{T}$ and $\mu = \lambda$, equations (2.1) and (2.2) imply

$$(|\widetilde{T}| - |\lambda|)|T|^{1/2}x_n \to 0 \text{ and } (|\widetilde{T}^*| - |\lambda|)|T|^{1/2}x_n \to 0.$$

Since $|\widetilde{T}| - |\lambda| \geq |T| - |\lambda| \geq |\widetilde{T}^*| - |\lambda|$, we have

$$(|T| - |\lambda|)|T|^{1/2}x_n \to 0,$$

and hence

$$|T|^2(|T|^2 - |\lambda|^2)x_n \to 0.$$

Now,

$$\begin{aligned} \|(|T|^2 - |\lambda|^2)x_n\|^2 &= ((|T|^2 - |\lambda|^2)x_n, |T|^2x_n) - ((|T|^2 - |\lambda|^2)x_n, |\lambda|^2x_n) \\ &= (|T|^2(|T|^2 - |\lambda|^2)x_n, x_n) - |\lambda|^2\|Tx_n\|^2 + |\lambda|^4 \\ &\to 0 - |\lambda|^4 + |\lambda|^4 = 0, \end{aligned}$$

implies $(|T|^2 - |\lambda|^2)x_n \to 0$. Setting $S = T$ and $\mu = \lambda$, equation (2.1) implies $(T^* - \overline{\lambda})x_n \to 0$. and the proof is complete. □

Theorem 3. *Let T be class A, $\lambda \neq 0$ and $\{x_n\}$ be a sequence of unit vectors. If $(T - \lambda)x_n \to 0$, then $(T^* - \overline{\lambda})x_n \to 0$.*

Proof. Again, the assumption implies $\|Tx_n\| \to |\lambda|$. Since T^2 is w-hyponormal by Theorem 1, and since

$$(T^2 - \lambda^2)x_n = (T+\lambda)(T-\lambda)x_n \to 0,$$

it follows from Theorem 2 that

$$(T^{2^*} - \overline{\lambda}^2)x_n \to 0.$$

Setting $S = T^2$ and $\mu = \lambda^2$, equation (2.1) implies $(|T^2| - |\lambda|^2)x_n \to 0$, and hence $\|\,|T^2|x_n\| \to |\lambda|^2$. Now,

$$\begin{aligned} \|(|T^2| - |T|^2)^{1/2}x_n\|^2 &= (|T^2|x_n, x_n) - (|T|^2x_n, x_n) \\ &\leq \|\,|T^2|x_n\| - \|Tx_n\|^2 \\ &\to |\lambda|^2 - |\lambda|^2 = 0, \end{aligned}$$

implies $(|T^2| - |T|^2)x_n \to 0$, and therefore

$$(|T|^2 - |\lambda|^2)x_n = (|T^2| - |\lambda|^2)x_n - (|T^2| - |T|^2)x_n \to 0.$$

Finally, setting $S = T$ and $\mu = \lambda$, equation (2.1) implies $(T^* - \overline{\lambda})x_n \to 0$. $\square$

Corollary 1. *If T is class A, then $\sigma_{na}(T)\backslash\{0\} = \sigma_a(T)\backslash\{0\}$.*

Corollary 2. *Let $T = U|T|$ be the polra decomposition of a class A operator T, $\lambda = |\lambda|e^{i\theta} \neq 0$ and $\{x_n\}$ be a sequence of unit vectors. If $(T-\lambda)x_n \to 0$, then $(|T| - |\lambda|)x_n \to 0$, $(|T^*| - |\lambda|)x_n \to 0$,$(U - e^{i\theta})x_n \to 0$ and $(U^* - e^{-i\theta})x_n \to 0$. Consequently, if $\lambda \in \sigma_a(T)$, then $|\lambda| \in \sigma_a(|T|) \cap \sigma_a(|T^*|)$ and $e^{i\theta} \in \sigma_{na}(U)$.*

Corollary 3. *Let T be p-hyponormal, $\lambda \in \mathbb{C}$ and $\{x_n\}$ be a sequence of unit vectors. If $(T-\lambda)x_n \to 0$, then $(T^* - \overline{\lambda})x_n \to 0$.*

3. The normality

If T is a hyponoprmal operator, then

$$\pi\|T^*T - TT^*\| \leq m_2(\sigma(T)),$$

where $m_2(\sigma(T))$ is the planar Lebesgue measure of the spectrum of T.

This was proved in Putnam [**Pu**] and is well known as Putnam's inequality. Let $\frac{1}{2} \leq p$. D.Xia [**Xi**] proved that the similar inequality holds for p-hyponormal : If T is p-hyponormal, then

$$\|(T^*T)^p - (TT^*)^p\| \leq \frac{p}{\pi} \iint_{\sigma(T)} \rho^{2p-1} d\rho d\theta \tag{3.1}$$

Also, M.Cho and M. Itoh [**ChI**] proved that a p-hyponormal operator T for ($0 < p < \frac{1}{2}$) satisfies the same inequality (3.1).

The results due to Putnam states as follows :

(i) If T is hyponormal or p-hyponormal $(0 < p < 1)$ and $m_2(\sigma(T)) = 0$, then T is normal.

(ii) A hyponormal or a p-hyponormal $(0 < p < 1)$*operator* is self-adjoint of $\sigma(T) \subset \mathbb{R}$; positive if $\sigma(T) \subset [0, \infty)$; or unitary if $\sigma(T) \subset \mathbb{T}$, the unit circle.

Putnam's result is known to hold for w-hyponormal operators T with the kernel condition $\ker T \subset \ker T^*$ [**AlW2**]. In this section, we will generalize Putnam's result to class A operators.

For w-hyponormal operators T with $\ker T \subset \ker T^*$, it was shown in [**AlW1**] that if $\widetilde{T}$ is normal, then T is normal. In the next theorem, we improve this result by dropping the kernel condition.

Theorem 4. *Let T be w-hyponormal. If $\widetilde{T}$ is normal, then T is normal.*

Theorem 5. *If T is w-hyponormal and $m_2(\sigma(T)) = 0$, then T is normal.*

Theorem 6 ([**IY** , Corollary 7]). *If both T and T^* are class A, then T is normal.*

Theorem 7. *If T is class A and $m_2(\sigma(T)) = 0$, then T is normal.*

Proof Since $\sigma(T^2) = \{z^2 : z \in \sigma(T)\}$, the assumption implies T^2 is w-hyponormal and $m_2(\sigma(T^2)) = 0$. Thus, T^2 is normal by Theorem 5. Since T is class A if and only if $(|T^*||T|^2|T^*|)^{1/2} \geq |T^*|^2$, if follows from ([**IY**, Theorem 1]) that $|T|^2 \geq (|T||T^*|^2|T|)^{1/2}$. The latter inequality holds if and only if the inequality $|T^{*^2}| \leq |T^*|^2$ holds. Consequently, the assumption that T is class A imply

$$|T|^2 \leq |T^2| = |T^{*^2}| \leq |T^*|^2,$$

and hence T^* is hyponormal. The result follows from Theorem 6. □

Corollary 4. *Let T be class A. (a) If $\sigma(T) \subset \mathbb{R}$, then T is self-adjoint. (b) If $\sigma(T) \subset [0, \infty)$, then T is positive. (c) If $\sigma(T) \subset \mathbb{T}$, then T is unitary.*

Corollary 5. *If T is a compact class A operator, then T is normal.*

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