Trends in Mathematics
Information Center for Mathematical Sciences
Volume 6, Number 2,December 2003, Pages 105–111

ALMOST AUTOMORPHISMS ON C^*-ALGEBRAS

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Abstract. This paper is a survey on almost automorphisms on unital C^*-algebras.

We introduce a recent result on the linear mapping in a Banach module over a unital C^*-algebra, and its applications to investigate automorphisms on unital C^*-algebras.

1. Introduction and Preliminaries

In 1940, S.M. Ulam [14] raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?

Let X and Y be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Hyers [5] showed that if $\epsilon > 0$ and $f : X \to Y$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \epsilon$$

for all $x \in X$.

Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\epsilon \ge 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in X$. Th.M. Rassias [13] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{2\epsilon}{2 - 2^p}||x||^p$$

for all $x \in X$. Găvruta [4] generalized the Rassias' result.

2000 *Mathematics Subject Classification.* Primary 46L05, 47B48, 39B52.

Key words and phrases. almost automorphism, C^*-algebra of real rank 0, stability, linear mapping.

This work was supported by Korea Research Foundation Grant KRF-2002-041-C00014.

Theorem 1.1 [4]**.** *Let G be an abelian group and Y a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a function such that*

$$\widetilde{\varphi}(x, y) = \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in G$. Suppose that $f : G \to Y$ is a mapping satisfying

$$\|f(x + y) - f(x) - f(y)\| \le \varphi(x, y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{1}{2} \widetilde{\varphi}(x, x)$$

for all $x \in G$.

For ring homomorphisms, D.G. Bourgin [2] proved the following.

Theorem 1.5 [2]**.** *Let ϵ and δ be nonnegative real numbers. Then every mapping f of a unital Banach algebra $\mathcal{A}$ onto a unital Banach algebra $\mathcal{B}$, satisfying*

$$\|f(x + y) - f(x) - f(y)\| \le \epsilon, \tag{1.1}$$

$$\|f(xy) - f(x)f(y)\| \le \delta \tag{1.2}$$

for all $x, y \in \mathcal{A}$, is a ring homomorphism.

In [1], Badora gave a simple proof of the following generalization of the Bourgin's result.

Theorem 1.6 [1, Theorem 1]**.** *Let $\mathcal{R}$ be a ring and $\mathcal{B}$ a Banach algebra. Let ϵ and δ be nonnegative real numbers. Assume that $f : \mathcal{R} \to \mathcal{B}$ satisfies* (1.1) and (1.2). Then there exists a unique ring homomorphism $h : \mathcal{R} \to \mathcal{B}$ such that

$$\|f(x) - h(x)\| \le \epsilon$$

for all $x \in \mathcal{R}$.

B.E. Johnson [6, Theorem 7.2] also investigated almost algebra $*$-homomorphisms between Banach $*$-algebras : Suppose that $\mathcal{U}$ and $\mathcal{B}$ are Banach $*$-algebras which satisfy the conditions of [6, Theorem 3.1]. Then for each positive ϵ and K there is a positive δ such that if $T \in L(\mathcal{U}, \mathcal{B})$ with $\|T\| < K$, $\|T^\vee\| < \delta$ and $\|T(x^*)^* - T(x)\| \le \delta \|x\| \quad (x \in \mathcal{U})$ then there is a $*$-homomorphism $T' : \mathcal{U} \to \mathcal{B}$ with $\|T - T'\| < \epsilon$. Here $L(\mathcal{U}, \mathcal{B})$ is the space of bounded linear maps from $\mathcal{U}$ into $\mathcal{B}$, and $T^\vee(x, y) = T(xy) - T(x)T(y) \quad (x, y \in \mathcal{U})$. See [6] for details.

The present paper is devoted to almost automorphisms on unital C^*-algebras and its aim is to present in a more or less organic form the great number of results on the subject published in the recent years.

In Section 2, we are going to introduce a recent result on the linear mapping in a Banach module over a unital C^*-algebra.

In Section 3, we are going to introduce recent results on automorphisms on unital C^*-algebras.

2. The linear mapping in a Banach module over a C^*-algebra

C. Park and H. Wee [12] extended the Găvruta's result to a Banach module over a unital C^*-algebra, i.e., they proved the generalized Hyers-Ulam-Rassias stability of the linear mapping in a Banach module over a unital C^*-algebra.

Theorem 2.1 [12, Theorem 1]**.** *Let A be a unital C^*-algebra with norm $|\cdot|$ and unitary group $\mathcal{U}(A)$, and ${}_AM$ a left A-module with norm $\|\cdot\|$. Let $f : {}_AM \to {}_AM$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : {}_AM \times {}_AM \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y) := \sum_{j=0}^{\infty} 2^{-j}\varphi(2^j x, 2^j y) < \infty, \tag{2.i}$$

$$\|uf(x + y) - f(ux) - f(uy)\| \le \varphi(x, y) \tag{2.ii}$$

for all $u \in \mathcal{U}(A)$ and all $x, y \in {}_AM$. Then there exists a unique A-linear mapping $T : {}_AM \to {}_AM$ such that

$$\|f(x) - T(x)\| \le \frac{1}{2}\widetilde{\varphi}(x, x) \tag{2.iii}$$

for all $x \in {}_AM$.

Sketch of the proof. Put $u = 1 \in \mathcal{U}(A)$. By the Găvruta result [4], there exists a unique additive mapping $T : {}_AM \to {}_AM$ satisfying (2.iii). The mapping $T : {}_AM \to {}_AM$ was given by $T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ for all $x \in {}_AM$.

By (2.ii), one can easily get

$$T(ux) = \lim_{n\to\infty} \frac{f(2^n ux)}{2^n} = \lim_{n\to\infty} \frac{uf(2^n x)}{2^n} = uT(x)$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_AM$.

Now let $a \in A$ $(a \neq 0)$ and N an integer greater than $4|a|$. Then

$$|\frac{a}{N}| = \frac{1}{N}|a| < \frac{|a|}{4|a|} = \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}.$$

By [7, Theorem 1], there exist three elements $u_1, u_2, u_3 \in \mathcal{U}(A)$ such that $3\frac{a}{N} = u_1 + u_2 + u_3$. Since T is additive, one can get

$$T(ax) = aT(x)$$

for all $x \in {}_AM$. Obviously, $T(0x) = 0T(x)$ for all $x \in {}_AM$. Hence

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in {}_AM$. So the unique additive mapping $T : {}_AM \to {}_AM$ is an A-linear mapping. □

This result is applied to investigate automorphisms on unital C^*-algebras, since each C^*-algebra can be considered as a module over a unital C^*-algebra $\mathbb{C}$.

3. Almost automorphisms on unital C^*-algebras

Throughout this section, let $\mathcal{B}$ be a unital C^*-algebra with norm $\|\cdot\|$ and unit e, $\mathcal{U}(\mathcal{B}) = \{u \in \mathcal{B} \mid uu^* = u^*u = e\}$, $\mathcal{B}_{sa} = \{x \in \mathcal{B} \mid x = x^*\}$, and r a real number greater than 1.

C. Park [10] investigated automorphisms on unital C^*-algebras.

Theorem 3.1 [10, Theorem 1]**.** *Let $h : \mathcal{B} \to \mathcal{B}$ be a bijective mapping satisfying $h(rx) = rh(x)$ for all $x \in \mathcal{B}$ for which there exists a function $\varphi : \mathcal{B}^4 \to [0,\infty)$ such that*

$$\widetilde{\varphi}(x,y,z,w) := \sum_{j=0}^{\infty} r^{-j}\varphi(r^j x, r^j y, r^j z, r^j w) < \infty, \tag{3.i}$$

(3.ii)
$$\|h(\mu x + \mu y + zw) - \mu h(x) - \mu h(y) - h(z)h(w)\| \le \varphi(x,y,z,w),$$

$$\|h(r^n u^*) - h(r^n u)^*\| \le \varphi(r^n u, r^n u, 0, 0) \tag{3.iii}$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid \ |\lambda| = 1\}$, all $u \in \mathcal{U}(\mathcal{B})$, all $n = 0,1,\cdots,$ and all $x,y,z,w \in \mathcal{B}$. Then the mapping $h : \mathcal{B} \to \mathcal{B}$ is an automorphism.

Sketch of the proof. Since $h(0) = rh(0)$, $h(0) = 0$. Put $z = w = 0$ and $\mu = 1 \in \mathbb{T}^1$ in (3.ii). By (3.ii) and the assumption that $h(rx) = rh(x)$ for all $x \in \mathcal{B}$,

$$\|h(x+y)-h(x)-h(y)\| = \frac{1}{r^n}\|h(r^n x+r^n y)-h(r^n x)-h(r^n y)\| \le \frac{1}{r^n}\varphi(r^n x, r^n y, 0, 0),$$

which tends to zero as $n \to \infty$ by (3.i). So

$$h(x+y) = h(x) + h(y)$$

for all $x,y \in \mathcal{B}$.

Similarly, one can obtain that

$$\begin{aligned} h(\mu x) &= \mu h(x), \\ h(zw) &= h(z)h(w), \\ h(u^*) &= h(u)^* \end{aligned}$$

for all $\mu \in \mathbb{T}^1$, all $x,z,w \in \mathcal{B}$, and all $u \in \mathcal{U}(\mathcal{B})$.

By the same reasoning as the proof of Theorem 2.1, the mapping $h : \mathcal{B} \to \mathcal{B}$ is a $\mathbb{C}$-linear mapping.

Since $h : \mathcal{B} \to \mathcal{B}$ is $\mathbb{C}$-linear and each $x \in \mathcal{B}$ is a finite linear combination of unitary elements (see [8, Theorem 4.1.7]), one can show that

$$h(x^*) = h(x)^*$$

for all $x \in \mathcal{B}$.

Therefore, the mapping $h : \mathcal{B} \to \mathcal{B}$ is an automorphism. □

Using the Rassias' technique [13, Theorem] and the method of the proof of Theorem 3.1, C. Park obtained the following.

Theorem 3.2 [10, Theorem 3]**.** *Let $h : \mathcal{B} \to \mathcal{B}$ be a bijective mapping satisfying $h(rx) = rh(x)$ for all $x \in \mathcal{B}$ for which there exists a function $\varphi : \mathcal{B}^4 \to [0, \infty)$ satisfying* (3.i) *and* (3.iii) *such that*

$$\|h(\mu x + \mu y + zw) - \mu h(x) - \mu h(y) - h(z)h(w)\| \le \varphi(x, y, z, w) \tag{3.iv}$$

for $\mu = 1, i$, and all $x, y, z, w \in \mathcal{B}$. If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{B}$, then the mapping $h : \mathcal{B} \to \mathcal{B}$ is an automorphism.

C. Park [9] proved the generalized Hyers-Ulam-Rassias stability of algebra $*$-homomorphisms on a C^*-algebra.

Theorem 3.3 [9, Theorem 2.1]**.** *Let $h : \mathcal{B} \to \mathcal{B}$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : \mathcal{B} \times \mathcal{B} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y) := \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x, 2^j y) < \infty, \tag{3.v}$$

$$\|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| \le \varphi(x, y), \tag{3.vi}$$

$$\|h(x^*) - h(x)^*\| \le \varphi(x, x), \tag{3.vii}$$

$$\|h(zw) - h(z)h(w)\| \le \varphi(z, w) \tag{3.viii}$$

for all $\mu \in \mathbb{T}^1$, all $z, w \in \mathcal{B}_{sa}$, and all $x, y \in \mathcal{B}$. Then there exists a unique algebra $*$-homomorphism $H : \mathcal{B} \to \mathcal{B}$ such that

$$\|h(x) - H(x)\| \le \frac{1}{2}\widetilde{\varphi}(x, x) \tag{3.ix}$$

for all $x \in \mathcal{B}$.

This result is applied to investigate automorphisms on a unital C^*-algebra.

Theorem 3.4 [9, Theorem 3.1]**.** *Let $h : \mathcal{B} \to \mathcal{B}$ be a bijective mapping satisfying $h(xy) = h(x)h(y)$ for all $x, y \in \mathcal{B}$ and $h(0) = 0$ for which there exists a function $\varphi : \mathcal{B} \times \mathcal{B} \to [0, \infty)$ satisfying* (3.v) *and* (3.vi) *such that*

$$\|h(2^n u^*) - h(2^n u)^*\| \le \varphi(2^n u, 2^n u) \tag{3.x}$$

for all $u \in \mathcal{U}(\mathcal{B})$ and all $n = 0, 1, \cdots$. Assume that $\lim_{n\to\infty} \frac{h(2^n e)}{2^n}$ is invertible. Then the bijective mapping $h : \mathcal{B} \to \mathcal{B}$ is an automorphism.

Under a weak condition, C. Park [11] proved the same result as Theorem 3.4.

Theorem 3.5 [11, Theorem 1]**.** *Let $h : \mathcal{B} \to \mathcal{B}$ be a bijective mapping satisfying $h(0) = 0$ and $h(2^n uy) = h(2^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{B})$, all $y \in \mathcal{B}$, and all $n = 0, 1, \cdots$, for which there exists a function $\varphi : \mathcal{B} \times \mathcal{B} \to [0, \infty)$ satisfying* (3.v), (3.vi) *and* (3.x). *Assume that $\lim_{n\to\infty} \frac{h(2^n e)}{2^n}$ is invertible. Then the bijective mapping $h : \mathcal{B} \to \mathcal{B}$ is an automorphism.*

Proof. We can consider the C^*-algebra $\mathcal{B}$ as a left $\mathbb{C}$-module. By Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{B} \to \mathcal{B}$ satisfying (3.ix). The $\mathbb{C}$-linear mapping $H : \mathcal{B} \to \mathcal{B}$ is given by

$$H(x) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n x) \tag{3.1}$$

for all $x \in \mathcal{B}$.

By (3.1) and (3.x), we get

$$H(u^*) = \lim_{n\to\infty} \frac{h(2^n u^*)}{2^n} = \lim_{n\to\infty} \frac{h(2^n u)^*}{2^n} = (\lim_{n\to\infty} \frac{h(2^n u)}{2^n})^* = H(u)^*$$

for all $u \in \mathcal{U}(\mathcal{B})$. By the same method as the proof of Theorem 3.1, one can show that

$$H(x^*) = H(x)^*$$

for all $x \in \mathcal{B}$.

Since $h(2^n uy) = h(2^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{B})$, all $y \in \mathcal{B}$, and all $n = 0, 1, \cdots$,

$$H(uy) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n uy) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n u)h(y) = H(u)h(y) \tag{3.2}$$

for all $u \in \mathcal{U}(\mathcal{B})$ and all $y \in \mathcal{B}$. By the additivity of H and (3.2),

$$2^n H(uy) = H(2^n uy) = H(u(2^n y)) = H(u)h(2^n y)$$

for all $u \in \mathcal{U}(\mathcal{B})$ and all $y \in \mathcal{B}$. Hence

$$H(uy) = \frac{1}{2^n} H(u)h(2^n y) = H(u)\frac{1}{2^n} h(2^n y) \tag{3.3}$$

for all $u \in \mathcal{U}(\mathcal{B})$ and all $y \in \mathcal{B}$. Taking the limit in (3.3) as $n \to \infty$, we obtain

$$H(uy) = H(u)H(y) \tag{3.4}$$

for all $u \in \mathcal{U}(\mathcal{B})$ and all $y \in \mathcal{B}$. Since H is $\mathbb{C}$-linear and each $x \in \mathcal{B}$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{B})$), it follows from (3.4) that

$$\begin{aligned} H(xy) &= H(\sum_{j=1}^m \lambda_j u_j y) = \sum_{j=1}^m \lambda_j H(u_j y) = \sum_{j=1}^m \lambda_j H(u_j)H(y) = H(\sum_{j=1}^m \lambda_j u_j)H(y) \\ &= H(x)H(y) \end{aligned}$$

for all $x, y \in \mathcal{B}$.

By (3.2) and (3.4),

$$H(e)H(y) = H(ey) = H(e)h(y)$$

for all $y \in \mathcal{B}$. Since $\lim_{n\to\infty} \frac{h(2^n e)}{2^n} = H(e)$ is invertible,

$$H(y) = h(y)$$

for all $y \in \mathcal{B}$.

Therefore, the bijective mapping $h : \mathcal{B} \to \mathcal{B}$ is an automorphism. □

Using a similar method to the proof of Theorem 3.5, C. Park [11] investigated continuous automorphisms on unital C^*-algebras.

Theorem 3.6 [11, Theorem 4]**.** *Let $\mathcal{B}$ be a unital C^*-algebra of real rank zero. Let $h : \mathcal{B} \to \mathcal{B}$ be a continuous bijective mapping satisfying $h(0) = 0$ and $h(2^n uy) = h(2^n u)h(y)$ for all $u \in I_1(\mathcal{B}_{sa}) := \{v \in \mathcal{B}_{sa} \mid \|v\| = 1, v$ is invertible$\}$, all $y \in \mathcal{B}$, and all $n = 0, 1, 2, \cdots$, for which there exists a function $\varphi : \mathcal{B} \times \mathcal{B} \to [0, \infty)$ satisfying* (3.v)*,* (3.vi) *and* (3.x)*. Assume that $\lim_{n\to\infty} \frac{h(2^n e)}{2^n}$ is invertible. Then the bijective mapping $h : \mathcal{B} \to \mathcal{B}$ is an automorphism.*

The above results will be applied to investigate automorphisms on Poisson C^*-algebras, automorphisms on Lie JC^*-algebras, and automorphisms on Poisson JC^*-algebras.

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