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WEYL'S THEOREM AND TENSOR PRODUCT FOR OPERATORS

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ABSTRACT. An operator T is called (p,k)-quasihyponormal if $T^{*k}(|T|^{2p} - |T^*|^{2p})T^k \geq 0$, $(0 < p \leq 1\ ;\ k \in \mathbb{Z}^+)$, which is a common generalization of p-quasihyponormality and k-quasihyponormality. In this paper we consider the Putnam's inequality, the Berger-Shaw's inequality, the Weyl's theorem and the tensor product for (p,k)-quasihyponormal operators.

Throughout this paper let $\mathcal{H}$ denote an infinite dimensional separable complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$ and let $\mathcal{K}(\mathcal{H})$ denote the closed ideal of all compact operators on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called *Fredholm*, denoted by $T \in \mathcal{F}$, if $\text{ran}(T)$ is closed and both $\ker(T)$ and $\mathcal{H}/\text{ran}(T)$ are finite dimensional. The *index* of a Fredholm operator $T \in \mathcal{B}(\mathcal{H})$, denoted by $\text{ind}(T)$, is given by

$$\text{ind}(T) = \dim \ker(T) - \dim (\mathcal{H}/ \text{ ran}(T)).$$

An operator $T \in \mathcal{B}(\mathcal{H})$ is called *Weyl*, denoted by $T \in \mathcal{F}_0$, if it is Fredholm of index zero. The point spectrum $\sigma_p(T)$, the essential spectrum $\sigma_e(T)$ and the Weyl spectrum $w(T)$ of $T \in \mathcal{B}(\mathcal{H})$ are defined by

$$\begin{aligned} \sigma_p(T) &:= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not ono to one}\}; \\ \sigma_e(T) &:= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}; \\ w(T) &:= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\} : \end{aligned}$$

evidently

$$\sigma_e(T) \subseteq w(T) \subseteq \sigma_e(T) \cup \text{acc}\sigma(T),$$

where we write accK for the *accumulation points* of $K \subseteq \mathbb{C}$. We write isoK := K $\setminus$ accK and

$$\pi_{00}(T) := \{\lambda \in \text{iso}\sigma(T) : 0 < \dim(\ker(T - \lambda)) < \infty\}$$

for the isolated eigenvalues of finite multiplicity. According to Corburn [3], we say that Weyl's theorem holds for $T \in \mathcal{B}(\mathcal{H})$ if

$$\sigma(T) \setminus w(T) = \pi_{00}(T).$$

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For p such as $0 < p \leq 1$, an operator $T \in \mathcal{B}(\mathcal{H})$ is called *p-hyponormal* if $(T^*T)^p - (TT^*)^p \geq 0$, and is called *$(p,k)$-quasihyponormal* if ${T^*}^k(|T|^{2p} - |T^*|^{2p})T^k \geq 0$, where $0 < p \leq 1$ and k is a positive integer. Especially, when $p = 1, k = 1$ and $p = k = 1$, T is called *k-quasihyponormal*, *p-quasihyponormal* and *quasihyponormal*, respectively. It is clear that

$$\begin{aligned}\{\text{hyponormal operators}\} &\subseteq \{p - \text{hyponormal operators}\} \\ &\subseteq \{p - \text{quasihyponormal operators}\} \\ &\subseteq \{(p,k) - \text{quasihyponormal operators}\}.\end{aligned}$$

and

$$\begin{aligned}\{\text{hyponormal operators}\} &\subseteq \{k - \text{quasihyponormal operators}\} \\ &\subseteq \{(p,k) - \text{quasihyponormal operators}\}.\end{aligned}$$

In what follows, all 2×2 block matrices are to be considered as a representation of an operator with respect to a pair of complementary, orthogonal subspaces of the underlying Hilbert space $\mathcal{H}$

Proposition 1. *If T is (p,k)-quasihyponormal operator, then T has the following matrix representation:*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \tag{1.1}$$

where T_1 is p-hyponormal on $\overline{ran(T^k)}$ and ${T_3}^k = 0$. Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.

From proposition 1 it follows that if T is (p,k)-quasihyponormal and $\overline{\text{ran}(T)} = \mathcal{H}$, then T is p-hyponormal. It should be noted that (1.1) is not a canonical form in that a given T can be represented in serveral ways. For example, if S is the unilateral shift, then

$$\begin{pmatrix} S & I - SS^* \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}$$

are unitarily equivalent and hyponormal.

Theorem 2. *If T is a (p,k)-quasihyponormal and the restriction T_1 of T on $\overline{ran(T^k)}$ is invertible, then T is similar to a direct sum of a p-hyponormal operator and a nilpotent operator.*

Theorem 3. *If T is a (p,k)-quasihyponormal and λ_0 is an isolated point of $\sigma(T)$ then λ_0 is an eigenvalue, i.e., T is isoloid.*

For some operators, there is an intimate relationship between the plannar Lebesgue measure of its spectrum and its self-commutator. For example, Putnam [Pu] obtained the norm estimation for the self-commutator of a hyponormal operator, called Putnam's inequality. This inequality is extended for a p-hyponormal operator by Xia [X], Cho-Itoh [CI] and Duggal [D2]. Also, this is extended for a p-quasihyponormal operator by Uchiyama [U2]. On the other hand, Berger-Shaw [BS] showed the trace norm estimation for the self-commutator of n-multicyclic hyponormal operator, called Berger-Shaw's inequality. This is extended for a p-hyponormal and p-quasihyponormal operator by Uchiyama [U1, U2].

Putnam's Inequality. *For a hyponormal operator* T, $||T^*T - TT^*|| \leq \frac{1}{\pi} Area(\sigma(T))$, *where Area means the planar Lebesgue measure.*

Berger-Shaw's Inequality. *If* T *is a* n*-multicyclic hyponormal operator, then* $[T^*, T] = T^*T - TT^*$ *is in the trace class, and* $tr([T^*, T]) \leq \frac{n}{\pi} Area(\sigma(T))$, *where Area means the planar Lebesgue measure.*

The following theorem is an extention of Putnam's inequality to the case of (p, k)-quasihyponormal operators.

Theorem 4. *If* T *is a* (p, k)*-quasihyponormal operator, then*

$$\|P\{(T^*T)^p - (TT^*)^p\} P\| \leq \min\left\{\frac{p}{\pi}\int_{\sigma(T)} r^{2p-1} dr d\theta, \quad \left(\frac{1}{\pi}\int_{\sigma(T)} r dr d\theta\right)^p\right\},$$

where P *is the projection onto* $\overline{ran(T^k)}$.

Corollary 5. *If* T *is a* (p, k)*-quasihyponormal operator and* $\sigma(T)$ *is Lebesgue null-set, then* T *is the direct sum of normal operator and nilpotent operator.*

For $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{R}(\sigma(T))$ denotes the set of all rational functions being analytic on $\sigma(T)$. The operator T is said to be *n-multicyclic* if there are n vectors $x_1, \cdots, x_n \in \mathcal{H}$, called generating vectors, such that $\bigvee\{g(T)x_i \mid i = 1, \cdots, n \quad \text{and} \quad g \in \mathcal{R}(\sigma(T))\} = \mathcal{H}$.

The following theorem is an extension of Berger-Shaw's inequality to the case of (p, k)-quasihyponormal operators.

Theorem 6. *If* T *is an* n*-multicyclic* (p, k)*-quasihyponormal operator, then we have :*

(i) *The restriction* T_1 *of* T *on* $\overline{ran(T^k)}$ *is also an* n*-multcyclic operator ;*

(ii) $\{P(T^*T)^p P - P(TT^*)^p P\}^{\frac{1}{p}}$ *belongs to the Schatten* $\frac{1}{p}$*-class and*

$$tr\left(\{P(T^*T)^p P - P(TT^*)^p P\}^{\frac{1}{p}}\right) \leq \frac{n}{\pi} Area(\sigma(T)),$$

where P *is the projection onto* $\overline{ran(T^k)}$.

S.H. Lee and W.Y. Lee [LL] showed that for hyponormal operators, the Weyl spectrum obeies the spectral mapping theorem and J.C. Hou and X.L. Zhang [HZ] showed that the Weyl spectral mapping theorem holds for quasihyponormal operators. We consider the Weyl spectral mapping theorem for (p, k)-quasihyponormal operators.

Theorem 7. *If* T *is* (p, k)*-quasihyponormal, then* $f(w(T)) = w(f(T))$ *for any analytic function* f *on a neighborhood of* $\sigma(T)$.

Weyl's theorem may or may not hold for a direct sum of operators for which Weyl's theorem holds. For example, if S is the unilateral shift on l_2, then Weyl's theorem holds for both S and S^*, while it does not hold for $S \oplus S^*$. But it was known [L1] that if A and B are isoloid and if Weyl's theorem holds for A and B then

$$\text{Weyl's theorem holds for } \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \Leftrightarrow w\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = w(A) \cup w(B).$$

In general, Weyl's theorem does not hold for operator matrix $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ even though Weyl's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ (see [L2]). But W.Y. Lee [L2] gave the passage from $w(A) \cup w(B)$ to $w\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and showed that if either $SP(A)$ or $SP(B)$ has no pseudoholes and if A is an isoloid operator for which Weyl's theorem holds then for every $C \in \mathcal{B}(\mathcal{H})$, then

$$\text{Weyl's theorem holds for } \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \Rightarrow \text{ Weyl's theorem holds for } \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

L.A. Corburn [Co], M. Cho-M. Itoh-S. Oshio [CIO], S.L. Campbell-B.C. Gupta [CG] and A. Uchiyama-S.V. Djordjevic [UD] showed that Weyl's theorem holds for hyponormal operators, p-hyponormal operators, k-quasihyponormal operators and p-quasihyponormal operators, respectively. We consider the Weyl's theorem for (p, k)-quasihyponormal operators. To do this we need the concept of the "spectral picture"[Pe] of the operator $T \in \mathcal{B}(\mathcal{H})$, denoted by $SP(T)$, which consists of the set $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$, and the indices associated with these holes and pseudoholes.

We have the following result.

Theorem 8. *Weyl's theorem holds for every (p, k)-quasihyponormal operators.*

The operation of taking tensor products $A \otimes B$ preserves many a property of $A, B \in \mathcal{B}(\mathcal{H})$, by no means all of them. Thus, whereas the normaloid property is invariant under tensor products, the spectraloid is not. Again, whereas $A \otimes B$ is normal if and only if A and B are [Ho], there exist paranormal operators A and B such that $A \otimes B$ is not paranormal [Sa]. On the other hand, J. Hou [Ho] and J. Stochel [St] showed that $T \otimes S$ is hyponormal on $\mathcal{H} \otimes \mathcal{H}$ if and only if each of T and S is hyponormal. More recently, B.P. Duggal [D1] demonstrated that the Hou–Stochel theorem remains true when one substitutes the term "p-hyponormal" for "hyponormal". Very recently, in [FK], it was shown that Hou–Stochel theorem remains true when one substitutes the term "p-quasihyponormal or w-hyponormal" for "hyponormal".

The next theorem extends J. Hou[Ho, Theorem 1.4] and Farenick and Kim[FK, Theorem 9].

Theorem 9. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ are nonzero operators. Then $A \otimes B$ is (p, k)-quasihyponormal if and only if one of the following holds:*

1. $A^k = 0$ *or* $B^k = 0$,
2. *A and B are (p,k)-quasihyponormal.*

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