Trends in Mathematics
Information Center for Mathematical Sciences
Volume 6, Number 2, December, 2003, Pages 51–57

EXPONENTS OF $r-$REGULAR PRIMITIVE MATRICES

MINGAI JIN, SANG GU LEE AND HAN GUK SEOL

ABSTRACT. A Boolean matrix A is said to be ***$r-$regular*** if each vertex in its digraph $G(A) = G$ has outdegree and indegree exactly r. A is primitive if and only if there exist minimum integer k such that $A^k > 0$. For such a matrix, its digraph G is strongly connected and given any ordered pair of vertices x and y there is a directed walk from x to y of length k. Using the graph theory, we determine the lower bounds and upper bounds of $r-$regular primitive matices in this paper. We also proved that the exponent of $r-$regular primitive tournament T is just 3 and the exponent of $r-$regular primitive symmetric matrix A satisfys $\exp(A) \leq 2(n-r)$.

1. BASIC CONCEPTS

Boolean matrix has a close relationship with the digraph. Many problems in digraph theory can be reformulated as Boolean matrix problems and we can deal with many problem of Boolean matrix by using the digraph theory. So it is worth to introduce the concepts on the Boolean matrix and digraph simultaneously.

Definition 1.1. *The **Boolean operations** $\{+, \cdot, ^c\}$ over set $\{0, 1\}$ as following:*

1) $0 + 0 = 0 \cdot 1 = 1 \cdot 0 = 0 \cdot 0 = 0$
2) $1 + 0 = 0 + 1 = 1 + 1 = 1 \cdot 1 = 1$
3) $1^c = 0, \quad 0^c = 1$

*and denote the set $\{0,\ 1\}$ with three calculations as above by β_0. Then $\{\beta_0, +, \cdot\}$ is the **Boolean algebra**.*

An $m \times n$ matrix over β_0 is called an $m \times n$ ***Boolean matrix***. The set of all $m \times n$ Boolean matrices is denoted by B_{mn}. If $m = n$, then B_{mn} can be written as B_n. And J is a matrix whose entries are all 1.

Definition 1.2. *Let $\{A^k : k = 1, 2, \cdots\}$ be the sequence of powers of matrix $A \in B_n (n \geq 2)$. If there is an integer m such that $A^m = J$, then the matrix A is called **primitive**. The smallest positive integer $k = \min\{m | A^m = J\}$ is called the **exponent** of A.*

Let $G = (V, E)$ be a strongly connected digraph of order n. Loops are permitted but no multiple arcs. If the greatest common divisor of its directed cycle lengths is equals to 1, then G is called ***primitive***. For any ordered pair of vertices x and y in such a digraph there is a directed walk from x to y of the smallest length k. Such a minimum integer k is called the ***exponent*** of G, and denoted by exp(G).

Let $G = (V, E)$ is a strongly connected digraph of order n and the vertex set of G is $V = \{a_1, a_2, \cdots, a_n\}$. Let a_{ij} be the multiplicity $m(a_i, a_j)$ of arcs of the form (a_i, a_j). Then the resulting matrix $A = A(G) = (a_{ij}), (i, j = 1, 2, \cdots, n)$ of order n is called the ***adjacency matrix*** of G. The matrix A characterizes G and it is easy to show that A is primitive if and only if G is primitive.

Conversely, let $A = (a_{ij}), (i, j = 1, 2, \cdots, n)$ be a Boolean matrix of order n, then there is a corresponding digraph $G = G(A)$ of order n as following. Let $V = \{a_1, a_2, \cdots, a_n\}$ be the vertex set and there is an arc $\alpha = (a_i, a_j)$ from a_i to a_j if and only if $a_{ij} = 1, (i, j = 1, 2, \cdots, n)$. In the event for a Boolean matrix A, the digraph G has no multiple arcs but loops are permitted.

After knowing the relationship between a Boolean matrix A of order n and the corresponding digraph G, we can deal with the problems of Boolean matrix by using the theory of graph.

2. The exponent of r-regular primitive matrices

Let $G = (V, E)$ denote a strongly connected digraph of order n. We permit loops but no multiple arcs. We first give some definitions and notations concerned in this section.

$N^-(u)$: inner neighborhood of vertex u. i.e. $N^-(u) = \{s | (s, u) \in E\}$

$N^+(u)$: outer neighborhood of vertex u. i.e. $N^+(u) = \{v | (u, v) \in E\}$

$d^-(u)$: indegree of vertex u. i.e. $d^-(u) = |\{s | (s, u) \in E\}| = |N^-(u)|$

$d^+(u)$: outdegree of vertex u. i.e. $d^+(u) = |\{v | (u, v) \in E\}| = |N^+(u)|$

$\delta^- = \min\{\text{indegree of vertices of } G\}$

$\delta^+ = \max\{\text{outdegree of vertices of } G\}$

$u \xrightarrow{k} v$: the directed walk of length k from u to v.

$\lfloor x \rfloor$: the least integer $\geq x$.

$\lceil x \rceil$: the most integer $\leq x$.

The ***distance*** from u to v, denoted by $d_G(u, v)$ or $d(u, v)$, is the length of the shortest directed walk from u to v. The ***diameter*** D of G is the maximum $d(u, v)$ among all ordered pairs $u, v \in V(G)$. The ***girth*** of G, denoted by g, is the length of a shortest cycle in G.

Let $G = (V, E)$ denote a $r-$regular primitive digraph of order n. Then the lower bound on exponent of $r-$regular primitive matices can be changed, according to the order of matrix.

For each vertex $v \in V$, we have $v \in N^+(v)$ or $v \notin N^+(v)$. When $r + 1 \leq n \leq r^2$ and $v \in N^+(v)$, then $|\{v\} \cup N^+(v)| = r$. So there exist some vertex $u \in V(G)$ such that the arc (v, u) not in E. Hence there is no walk $v \xrightarrow{1} u$. In the other hand, if $v \notin N^+(v)$ for any vertex $v \in V$, then there is no walk $v \xrightarrow{1} v$.

Suppose $r + 1 \leq n \leq 2r - 1$. Then we have $N^+(u) \cap N^-(v) \neq \emptyset$ for each pair $u, v \in V$. Hence there exist the walk $u \xrightarrow{2} v$ for each pair $u, v \in V$.

Let v be any vertex of G. Since G is $r-$regular primitive digraph, we have

$$|N^-(v)| = |N^+(v)| = r$$

Suppose $N^+(v) = \{v_1, v_2, \cdots, v_r\}$, then $|\cup_{i=1}^r N^+(v_i)| \leq r^2$. But the order of G is greater than r^2, hence there exist some vertex u such that

$$u \notin \cup_{i=1}^r N^+(v_i)$$

So there is no walk $v \xrightarrow{2} u$.

From the above, we have the following theorem.

Theorem 2.1. *Let $G = (V, E)$ be a $r-$regular primitive digraph of order n, then the exponent $\exp(G)$ of G satisfies following:*
1) $n = r \implies \exp(G) = 1$;
2) $r + 1 \leq n \leq 2r - 1 \implies \exp(G) = 2$;
3) $2r \leq n \leq r^2 \implies \exp(G) \geq 2$;
4) $n \geq 1 + r^2 \implies \exp(G) \geq 3$.

Theorem 2.2. *Let $G = (V, E)$ be a $r-$regular primitive digraph of order $n \geq 1+r^k$, then the exponent $\exp(G)$ of G satisfies*

$$\exp(G) \geq k + 1$$

Proof. Let v be any vertex of G. Since G is $r-$regular primitive, we have

$$|N^-(v)| = |N^+(v)| = r$$

Suppose $N^+(v) = \{v_1, v_2, \cdots, v_r\} = V_1$, then

$$|\cup_{i=1}^r N^+(v_i)| = |\cup_{u\in V_1} N^+(u)| \leq r^2$$

Denote $V_2 = \cup_{u\in V_1} N^+(u)$, then we can show that

$$|\cup_{u\in V_2} N^+(u)| \leq r^2 \cdot r = r^3$$

Denote $V_3 = \cup_{u\in V_2} N^+(u)$, then we also can show that

$$|\cup_{u\in V_3} N^+(u)| \leq r^3 \cdot r = r^4; \cdots$$

In such a way, we can show that

$$|\cup_{u\in V_{k-1}} N^+(u)| \leq r^{k-1} \cdot r = r^k$$

But the order of G is greater than r^k, so there exist some vertex u' such that

$$u' \notin \cup_{u\in V_{k-1}} N^+(u)$$

Hence there is no walk $v \xrightarrow{2} u'$. The proof is completed.

Corollary 2.1. *Let $G = (V, E)$ be a $3-$regular primitive digraph of order n, then the exponent $\exp(G)$ of G satisfies the following:*
1) $n = 3 \implies \exp(G) = 1$;
2) $n = 4, 5 \implies \exp(G) = 2$;
3) $6 \leq n \leq 9 \implies \exp(G) \geq 2$;
4) $n \geq 10 \implies \exp(G) \geq 3$

Corollary 2.2. *Let $G = (V, E)$ be a $3-$regular primitive digraph of order $n \geq 1+3^k$, then the exponent $\exp(G)$ of G satisfies*

$$\exp(G) \geq k + 1$$

As a $r-$regular primitive matrix A, the exponent of A depends only on its digraph. So it is necessary to study the characteristic of $r-$regular primitive digraph.

Let $G = (V, E)$ is a $r-$regular primitive digraph of order n and its diameter is D. $u, v \in G$ be two vertices with distance $d(u, v) = D$ with the associated directed walk

$$u \to u_1 \to u_2 \to \cdots \to u_{D-1} \to v$$

Then we have

$$\{u, u_1, u_2, \cdots, u_{D-2}\} \cap N^-(v) = \emptyset$$

Since $N^-(v) = r$, we can show that

$$(D - 1) + r = |\{u, u_1, u_2, \cdots, u_{D-2}\} \cup N^-(v)| \leq |V| = n$$

Hence we have following theorem

Theorem 2.3. *Suppose $G = (V, E)$ is a $r-$regular primitive digraph of order n and its diameter is D. Then*

$$D \leq n - r + 1$$

If the $r-$regular primitive digraph has loops, then we have the following theorem.

Theorem 2.4. *Suppose G is a $r-$regular primitive digraph of order n having $p \geq 1$ loops. Then*

$$\exp(G) \leq 2(n - r + 1) - p$$

Proof. Let W be the set of $p$vertices which are incident with a loop and $u, v \in G$ be any two vertices. Denote the distance from u to set W by

$$d(u, W) = \min\{d(u, w) | w \in W\}$$

then there must exist a directed walk from u to $w' \in W$ whose length is at most $d' = d(u, w') = d(u, W)$. We denote the walk as following

$$u \to u_1 \to u_2 \to \cdots \to u_{d'-1} \to w'$$

Then $[\{u, u_1, u_2, \cdots, u_{d'-2}\} \cup N^-(v)] \cap W = \emptyset$ and have

$$\{u, u_1, u_2, \cdots, u_{d'-2}\} \cap N^-(v) = \emptyset$$

Since $N^-(v) = r$, we can show that

$$(d' - 1) + r = |\{u, u_1, u_2, \cdots, u_{d'-2}\} \cup N^-(v)| \leq |V| - |W| = n - p$$

Hence

$$d' \leq n - r - p + 1$$

By the theorem 2.3, the directed walk from w' to v satisfies $d(w', u) \leq n - r + 1$. Combining these two directed walk we obtain a directed walk from u to v of length at most $2(n - r + 1) - p$. Taking advantage of the loop at vertex w' we can obtain a directed walk from u to v whose length is exactly

$$2(n - r + 1) - p$$

The proof is completed.

If the $r-$regular primitive digraph has no loops, then we have the following theorem.

Theorem 2.5. *Suppose G is a $r-$regular primitive digraph of order n with thr girth $g \geq 2$, then*

$$\exp(G) \leq (g + 1)(n - r - g + 1)$$

Lemma 2.1. *([2]) Suppose $G = (V, E)$ is a primitive digraph with diameter D and girth g. Then*

$$\exp(G) \leq D + 1 + g(D - 1)$$

Lemma 2.2. *([22]) Let $G = (V, E)$ be a digraph such that $d^+(x) = 3$ for every vertex x of G. Let n be the order of G and g be its girth. Then*

$$n \geq 3g - 2$$

From the lemma 2.1 and corollary 2.3 we have

Theorem 2.6. *Suppose $G = (V, E)$ is a $3-$regular primitive digraph of order $n \geq 2r$ with diameter D and girth g. Then*

$$\exp(D) \leq \frac{1}{3}(n^2 + 2n - 9)$$

3. Specil 3-regular primitive matrices

Definition 3.1. *A digraph obtained from the complete graph K_n by assigning a direction to each of its edges. If A is its adjacency matrix, then A is a $(0, 1)-$matrix satisfying:*

$$A + A^T = J - I$$

thus a matrix A is called a ***tournament matrix****.*

If digraph G is a strongly connected tournament of order $n \geq 5$, then we know that the exponent of G satisfy $3 \leq \exp(G) \leq n + 2$.

If digraph G is a strongly connected tournament of order n , then for any vertex u, we have

$$d^+(u) + d^-(u) = n - 1 \quad \text{and} \quad N^+(u) \cap N^-(u) = \emptyset$$

It means the order n of a $r-$regular primitive tournament G must be $2r+1$ with no loop in G. In the other hand, each vertex $u \in V$ satisfies

$$N^{+}(u) \cap N^{-}(u) = \emptyset \quad \text{and} \quad d^{+}(u) + d^{-}(u) = 2r$$

Let $N^{+}(u) = \{u_1, u_2, \cdots, u_r\}$, we show that

$$| \cup_{i=1}^{r} N^{+}(u_i)| \geq r+1$$

If $N^{-}(u) = \{u'_1, u'_2, \cdots, u'_r\}$, then we also can show that $| \cup_{i=1}^{r} N^{-}(u'_i)| \geq r+1$.

Let A be the adjacency Boolean matrix of G, then by the arbitrary of u, the line sum of A^2 must greater than $r+1$ and $\mathrm{tr}(A^2) = 0$. In the other hand, the line sum of A is r and $\mathrm{tr}(A) = 0$. Hence

$$A^2 A = A^3 = J$$

We obtain the following theorem.

Theorem 3.1. *If digraph $G = (V, E)$ is a $r-$regular primitive tournament of order n, then the exponent of G is* 3.

A matrix A of order n is called ***symmetric*** if $A = A^T$.

If matrix A is a symmetric primitive of order n, it is well known that the exponent of A satisfies $\exp(A) \leq 2n-2$. In this paper, we give the upper bound on exponent of $3-$regular primitive symmetric matrix as following theorem.

Theorem 3.2. *If digraph $G = (V, E)$ is a $r-$regular primitive symmetric of order $n \geq 2r$, then the exponent of G satisfies*

$$\exp(G) \leq 2(n-r)$$

where the the order r of regular satisfies $r \geq 3$.

References

[1] Brauldi and Ryser. *Combinatorial Matrix Theory.* Encyclopedia. Math. Appl. 1991.

[2] Shen, Jian. *Exponents of 2-regular digraphs.* Discrete Math. 214 (2000).

[3] Shao, Jia Yu. *Matrices permutation equivalent to primitive matrices.* Linear Algebra Appl. (1985).

[4] Liu, Bo Lian. *A note on the exponents of primitive (0,1) matrices.* Linear Algebra Appl. (1990).

[5] Liu, Bo Lian; McKay, Brendan D.; Wormald, Nicholas C.; Zhang, Ke Min. *The exponent set of symmetric primitive (0,1) matrices with zero trace.* Linear Algebra Appl. 133 (1990).

[6] Zhou Bo, Bolian Liu. *New Results on the Common Consequent Index of a Binary Relation.* Europ. J. Combinatorics (2000).

[7] B. Cheng, B. Liu. *Matrices of zeros and ones with the maximum jump number.* Linear Algebra Appl. (1998).

[8] BoLian Liu , Wen Jiang. *On a conjecture of Lewin's problem.* Linear Algebra Appl. (2001).

[9] Brualdi, R. A.; Shen, Jian. *Diameter of the NEPS of bipartite graphs.* Discrete Math.(2001).

[10] Shen, Jian. *On the girth of digraphs.* Discrete Math. 211 (2000).

[11] Shen, Jian; Gregory, D. A.. *Exponents of vertex-transitive digraphs.* Combinatorics and applications(Tianjin,1996). Discrete Math. 212(2000), no. 3.

[12] Shen, Jian.*Neufeld, Stewart; On a problem of Lewin*. Linear Algebra Appl. 274 (1998).
[13] Shen, Jian. *Proof of a conjecture about the exponent of primitive matrices*. Linear Algebra Appl. 216 (1995), 185–203.
[14] Jian Shen, Stewart Neufeld. *Directed Triangles in Digraphs*. Journal of Combinatorial Theory, Series B 74, (1998).
[15] Stewart Neufeld. *A Diameter Bound on The Exponent of a primitive Directed Graph*. Linear Algebra and Its Appl. 245: 27-47(1996)
[16] Shen, Jian. *On a problem of Lewin*. Linear Algebra Appl. 274 (1998), No 2, 405–407.
[17] Shen, Jian. *A Problem on the Exponent of Primitive Digraphs*. Linear Algebra Appl. (1996).
[18] Shen, Jian; *A bound on the exponent of primitivity in term of diameter*. Linear Algebra Appl. 244(1996) 21-34.
[19] Shen, Jian. *An improvement of the Dulmage-Mendelsohn theorem*. Discrete Math. 158(1-3)(1996) 295–297
[20] Gregory, D. A.; Kirkland, S. J.; Pullman, N. J. *A bound on the exponent of a primitive matrix using Boolean rank*. Linear Algebra Appl. 217 (1995), 101–116.
[21] C.T.Hoang and B. Reed. *A Note On Short Cycles In Digraphs*. Discrete Math. 66(1987) 103–107
[22] Yahya Ould Hamidoune. *A Note on Minimal Directed Graphs with Given Girth*. Journal of Combinatorial Theory, Series B 43, 343-348(1987).
[23] Jose Soares. *Maximum Diameter of Regular Digraphs*.

TianJin University of Finance & Economics
E-mail address: jma@math.skku.ac.kr

DEPARTMENT OF MATHEMATICS SUNGKYUNKWAN UNIVERSITY SUWON 440-746 K OREA
E-mail address: sglee@math.skku.ac.kr

DEPARTMENT OF MATHEMATICS DAEJIN UNIVERSITY POCHEON 487-711 KOREA
E-mail address: hgseol@skku.edu