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ENTANGLED STATES ARISING FROM INDECOMPOSABLE POSITIVE LINEAR MAPS

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ABSTRACT. We survey the duality theory between positive linear maps in matrix algebras and entanglements in block matrices, and review how to construct entangled states from examples of indecomposable positive linear maps between matrix algebras. We also give questions arising from these examples.

1. INTRODUCTION

Every linear functional on the matrix algebra M_n of all $n \times n$ matrices over the complex field is represented by a matrix with the same size by the bilinear pairing

$$\langle [a_{ij}], [b_{ij}] \rangle = \sum_{i,j} a_{ij} b_{ij}.$$

A linear functional is positive, that is, sends a positive semi-definite matrices into nonnegative numbers, if and only if the representing matrix is positive semi-definite. A linear functional is unital, that is, send the identity matrix to the number 1, if and only if the representing matrix is of trace one. In this sense, every state which is by definition a unital positive linear functional is represented by a density matrix, that is, a positive demi-definite matrix with the trace one.

From now on, we will pay attention on the block matrices, or equivalently tensor products of two matrices. We identify an $m \times n$ matrix $z \in M_{m\times n}$ and a vector $\widetilde{z} \in \mathbb{C}^n \otimes \mathbb{C}^m$ as follows: For $z = [z_{ik}] \in M_{m\times n}$, we define

$$(1) \qquad \begin{aligned} z_i &= \sum_{k=1}^{n} z_{ik} e_k \in \mathbb{C}^n, \qquad i = 1, 2, \dots, m, \\ \widetilde{z} &= \sum_{i=1}^{m} z_i \otimes e_i \in \mathbb{C}^n \otimes \mathbb{C}^m. \end{aligned}$$

Simply, $\widetilde{z}$ is the column vector whose i-th block of entries comes from the i-th row of the matrix z. Then $z \mapsto \widetilde{z}$ defines an inner product isomorphism from $M_{m\times n}$ onto $\mathbb{C}^n \otimes \mathbb{C}^m$. Every state, or density matrix A in $M_n \otimes M_m$ is the convex combination of the positive semi-definite rank one matrices $\widetilde{z}\,\widetilde{z}^* \in M_n \otimes M_m$ with range vectors $\widetilde{z}$. We say that a state or a density matrix is *separable* if it belongs to the convex cone generated by the set

$$\{\widetilde{z}\,\widetilde{z}^* \in M_n \otimes M_m : \text{ rank of } z = 1\}.$$

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For example, we consider the following 4×4 matrices

$$A = \frac{1}{2}\begin{pmatrix} e_{11} & e_{11} \\ e_{11} & e_{11} \end{pmatrix}, \qquad B = \frac{1}{2}\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix},$$

where e_{ij} denotes the usual 2×2 matrix units. We see that A is separable, since the range vector is associated with the rank one matrix $e_{11} + e_{21}$. But, B is not separable, since the range vector is associated with the matrix $e_{11} + e_{22}$ which is not of rank one. We say that a state or a density matrix is *entangled* if it is not separable. If the rank of a given density matrix is greater than 1, then it is very difficult to distinguish entangled states from separable ones. The theory of positive linear maps between matrix algebras is turned out to be very useful to distinguish entanglement, and it is a primary purpose of this survey article to explain this.

The notion of entanglement was originated from quantum physics, and has been playing a key rôle in the quantum information theory and quantum communication theory during the past decade (see [14] for a survey). On the other hand, properties of positive linear maps between matrix algebras have been studied by operator algebraists from the sixties [28]. It is now turned out that these two theories are dual each other. For example, the author [9] used various cones of block matrices adapting the idea of Woronowicz [34] to characterize the boundary structures of various kinds of positive linear maps between matrix algebras. One of these notions for block matrices is turned out to be nothing but entanglement. On the other hand, people in the quantum information theory used the notion of positive linear maps to characterize entanglement. See [13], [15] and [33] for examples. This duality was used [32] to construct indecomposable positive linear maps arising from entanglements. In this article, we explain the duality studied in [9] and outline how to construct a family of entangled states as was done in [12]. The survey part for the duality is partly an excerption from [12].

Throughout this article, every vector will be considered as a column vector. If $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$ then x will be considered as an $m \times 1$ matrix, and y^* will be considered as a $1 \times n$ matrix, and so xy^* is an $m \times n$ rank one matrix whose range is generated by x and whose kernel is orthogonal to y.

2. Duality and Construction

A linear map between C^*-algebras is said to be *positive* if it send every positive element to a positive element. A linear map $\phi : A \to B$ is said to be *s-positive* if the map

$$\phi_s : M_s(A) \to M_s(B) : [a_{ij}] \mapsto [\phi(a_{ij})]$$

is positive, where $M_s(A)$ is the C^*-algebra of all $s \times s$ matrices over A. We say that ϕ is *completely positive* if ϕ is s-positive for every $s = 1, 2, \dots$. The transpose map

$$X \mapsto X^{\mathrm{t}}, \qquad X \in M_n$$

is a typical example of a positive linear map which is not completely positive. We say that a linear map $\phi : M_m \to M_n$ is said to be *s-copositive* if $X \mapsto \phi(X^{\mathrm{t}})$ is s-positive, and *completely copositive* if it is s-copositive for every $s = 1, 2, \dots$.

The cone of all s-positive (respectively t-copositive) linear maps from A into B will be denoted by $\mathbb{P}_s[A,B]$ (respectively $\mathbb{P}^t[A,B]$), and just by $\mathbb{P}_s$ (respectively $\mathbb{P}^t$) whenever the domain and range are clear.

For a given $m\times n$ matrix $z\in M_{m\times n}$, we note that $\widetilde{z}\widetilde{z}^*$ belongs to $M_n\otimes M_m$, which is identified with the space $M_m(M_n)$ of all $m\times m$ matrices whose entries are $n\times n$ matrices, where $\widetilde{z}$ is defined as in (1). For $A\in M_n\otimes M_m$, we denote by A^τ the *block transpose* or *partial transpose* of A, that is,

$$\left(\sum_{i,j=1}^m a_{ij}\otimes e_{ij}\right)^\tau=\sum_{i,j=1}^m a_{ji}\otimes e_{ij}.$$

Now, we define

$$\begin{aligned}\mathbb{V}_s =&\operatorname{conv}\{\widetilde{z}\widetilde{z}^*\in M_n\otimes M_m:\ \text{rank of } z\le s\},\\ \mathbb{V}^s =&\operatorname{conv}\{(\widetilde{z}\widetilde{z}^*)^\tau\in M_n\otimes M_m:\ \text{rank of } z\le s\},\end{aligned}$$

for $s=1,2,\dots,m\wedge n$, where $\operatorname{conv} X$ means the convex set generated by X, and $m\wedge n$ denotes the minimum of m and n. It is clear that $\mathbb{V}_{m\wedge n}$ coincides with the cone of all positive semi-definite $mn\times mn$ matrices. It is easily seen that $\mathbb{V}_1=\mathbb{V}^1$. We also have the following chains

$$\mathbb{V}_1\subset\mathbb{V}_2\subset\cdots\subset\mathbb{V}_{m\wedge n},\qquad \mathbb{V}^1\subset\mathbb{V}^2\subset\cdots\subset\mathbb{V}^{m\wedge n}$$

of inclusions. We note that a density matrix $M_n\otimes M_m$ represents a *separable* state if and only if $A\in\mathbb{V}_1$. The minimum number s with $A\in\mathbb{V}_s$ is the *Schmidt number* of $A\in M_n\otimes M_m$, in the language of quantum information theory.

Motivated by the work of Woronowicz [34] (see also [16], [29]), we have considered in [9] the bi-linear pairing between $M_n\otimes M_m$ and the space $\mathcal{L}(M_m,M_n)$ of all linear maps from M_m into M_n, given by

$$\langle A,\phi\rangle=\operatorname{Tr}\left[\left(\sum_{i,j=1}^m\phi(e_{ij})\otimes e_{ij}\right)A^{\mathrm{t}}\right]=\sum_{i,j=1}^m\langle\phi(e_{ij}),a_{ij}\rangle,$$

for $A=\sum_{i,j=1}^m a_{ij}\otimes e_{ij}\in M_n\otimes M_m$ and $\phi\in\mathcal{L}(M_m,M_n)$, where the bi-linear form in the right-side is given by $\langle a,b\rangle=\operatorname{Tr}(ba^{\mathrm{t}})$ for $a,b\in M_n$. In this duality, the pairs

$$(\mathbb{V}_s,\ \mathbb{P}_s),\qquad(\mathbb{V}^t,\ \mathbb{P}^t),\qquad(\mathbb{V}_s\cap\mathbb{V}^t,\ \mathbb{P}_s+\mathbb{P}^t)$$

are dual each other, in the sense that $A\in\mathbb{V}_s$ (respectively $\phi\in\mathbb{P}_s$) if and only if $\langle A,\phi\rangle\ge 0$ for each $\phi\in\mathbb{P}_s$ (respectively $A\in\mathbb{V}_s$), and similarly for others.

A linear maps in the cone

$$\mathbb{D}:=\mathbb{P}_{m\wedge n}+\mathbb{P}^{m\wedge n}$$

is said to be *decomposable*, that is, a linear map is said to be decomposable if it is the sum of a completely positive linear map and a completely copositive linear map. Every decomposable map is positive, but the converse is not true. There are many examples of indecomposable positive linear maps in the literature [3], [5],

[10], [11], [17], [19], [25], [26], [29], [30], [31], [32]. Since the cone $\mathbb{V}_{m\wedge n}$ consists of all positive semi-definite matrices, the cone

$$\mathbb{T} := \mathbb{V}_{m\wedge n} \cap \mathbb{V}^{m\wedge n}$$

consists of all positive semi-definite matrices whose block transposes are also positive semi-definite, or positive semi-definite matrices with *positive partial transposes* in the language of quantum information theory. The duality between two cones $\mathbb{D}$ and $\mathbb{T}$ is summarized by

$$\begin{aligned} A \in \mathbb{T} &\iff \langle A, \phi\rangle \ge 0 \text{ for each } \phi \in \mathbb{D}, \\ \phi \in \mathbb{D} &\iff \langle A, \phi\rangle \ge 0 \text{ for each } A \in \mathbb{T}, \end{aligned}$$

for $A \in M_n \otimes M_m$ and $\phi \in \mathcal{L}(M_m, M_n)$.

For subsets $X \subset \mathbb{D}$ and $Y \subset \mathbb{T}$, we define $X' \subset \mathbb{T}$ by

$$X' = \{A \in \mathbb{T} : \langle A, \phi\rangle = 0 \text{ for every } \phi \in X\},$$

and $Y' \subset \mathbb{D}$ similarly. It is easy to see that X' is a face of $\mathbb{T}$, and every exposed face of $\mathbb{T}$ arises in this way. We also note that if $\phi \in \mathbb{D}$ is an interior point of a face F of $\mathbb{D}$ then $F' = \{\phi\}'$. The set $\{\phi\}'$ will be written by ϕ'. Dual results also hold, of course. This kind of duality holds for much more general situations, and was used to characterize maximal faces of the cones $\mathbb{P}_s[M_m, M_n]$ for $s = 1, 2, \dots, m \wedge n$ [20], [21].

Next, we review the intrinsic characterization of faces of the cone $\mathbb{D}$ as was in [24], which was motivated to find out all faces of the cone $\mathbb{P}_1[M_2, M_2]$ of all positive linear maps between 2×2 matrices [1], [23]. For a finite subset $\mathcal{V} = \{V_1, V_2, \dots, V_\nu\}$ of $M_{m\times n}$, we define the linear maps $\phi_{\mathcal{V}}$ and $\phi^{\mathcal{V}}$ from M_m into M_n by

$$\phi_{\mathcal{V}} : X \mapsto \sum_{i=1}^{\nu} V_i^* X V_i, \qquad \phi^{\mathcal{V}} : X \mapsto \sum_{i=1}^{\nu} V_i^* X^{\mathrm{t}} V_i, \qquad X \in M_m.$$

We also denote by $\phi_V = \phi_{\{V\}}$ and $\phi^V = \phi^{\{V\}}$. It is well-known [4], [18] that every completely positive (respectively completely copositive) linear map from M_m into M_n is of the form $\phi_{\mathcal{V}}$ (respectively $\phi^{\mathcal{V}}$). Every linear map $\phi : M_m \to M_n$ corresponds to a block matrix in $M_n \otimes M_m = M_m(M_n)$ by

$$\phi \mapsto [\phi(e_{ij})]_{i,j=1,2,\dots,m}. \tag{2}$$

It is also well known [4] that a linear map $\phi : M_m \to M_n$ is completely positive if and only if it is $(m \wedge n)$-positive if and only if the corresponding matrix in (2) is positive semi-definite. For a subspace E of $M_{m\times n}$, we define

$$\begin{aligned} \Phi_E &= \{\phi_{\mathcal{V}} \in \mathbb{P}_{m\wedge n}[M_m, M_n] : \operatorname{span}\mathcal{V} \subset E\} \\ \Phi^E &= \{\phi^{\mathcal{V}} \in \mathbb{P}^{m\wedge n}[M_m, M_n] : \operatorname{span}\mathcal{V} \subset E\}, \end{aligned} \tag{3}$$

where $\operatorname{span}\mathcal{V}$ denotes the span of the set $\mathcal{V}$. We have shown in [22] that the correspondence $E \mapsto \Phi_E$ gives rise to a lattice isomorphism from the lattice of all subspaces of the vector space $M_{m\times n}$ onto the lattice of all faces of the convex cone $\mathbb{P}_{m\wedge n}[M_m, M_n]$. Of course, the same result holds for the map $E \mapsto \Phi^E$.

Let C be the convex hull of the cones C_1 and C_2. If F is a face of C then it is easy to see that $F \cap C_1$ and $F \cap C_2$ are faces of C_1 and C_2, respectively, and F is the convex hull of $F \cap C_1$ and $F \cap C_2$. This is immediately applied to characterize faces of the cone $\mathbb{D}$ which is the convex hull of the cones $\mathbb{P}_{m\wedge n}$ and $\mathbb{P}^{m\wedge n}$. For a given face F of $\mathbb{D}$, we see that $F \cap \mathbb{P}_{m\wedge n}$ is a face of $\mathbb{P}_{m\wedge n}$, and so it is of the form Φ_D for a subspace D of $M_{m\times n}$. Similarly, $F \cap \mathbb{P}^{m\wedge n} = \Phi^E$ for a subspace E of $M_{m\times n}$. Therefore, we see that every face of $\mathbb{D}$ is of the form

$$\sigma(D,E) := \operatorname{conv}\{\Phi_D, \Phi^E\}, \tag{4}$$

for subspace D and E of $M_{m\times n}$. If we assume the following condition

$$\sigma(D,E) \cap \mathbb{P}_{m\wedge n} = \Phi_D, \qquad \sigma(D,E) \cap \mathbb{P}^{m\wedge n} = \Phi^E$$

then it is clear that every face of $\mathbb{D}$ is uniquely expressed as in (4). It seems to be very difficult in general to determine all the pairs of subspaces which give rise to faces of $\mathbb{D}$. See [1] for the simplest case $m = n = 2$.

Suppose that we are given an example of an indecomposable positive linear map $\phi : M_m \to M_n$. If we define

$$\alpha = \sup\{t \in \mathbb{R} : \phi_t := (1-t)\mathrm{Tr} + t\phi \in \mathbb{D}\},$$

then we see that

$$\phi_\alpha := (1-\alpha)\mathrm{Tr} + \alpha\phi$$

is a boundary point of the cone $\mathbb{D}$, but is an interior point of the cone $\mathbb{P}_1$ of all positive linear maps. We recall that a convex set is partitioned into the family of interiors of the faces. Therefore, the boundary point ϕ_α determines a proper face $\sigma(D,E)$ of $\mathbb{D}$ whose interior contains ϕ_α. Since $\sigma(D,E)$ is a convex subset of $\mathbb{P}_1$, we have the two cases:

$$\operatorname{int}\sigma(D,E) \subset \operatorname{int}\mathbb{P}_1 \qquad \text{or} \qquad \sigma(D,E) \subset \partial\mathbb{P}_1.$$

The above construction gives us a face $\sigma(D,E)$ of $\mathbb{D}$ whose interior is contained in the interior of $\mathbb{P}_1$. We refer to [12] for a simple proof for the following:

Theorem 2.1. *If (D,E) is a pair of spaces of $m \times n$ matrices which gives rise to a proper face $\sigma(D,E)$ of $\mathbb{D}$ with $\operatorname{int}\sigma(D,E) \subset \operatorname{int}\mathbb{P}_1$ then every nonzero element A of the face $\sigma(D,E)'$ of $\mathbb{T}$ belongs to $\mathbb{T} \setminus \mathbb{V}_1$.*

3. Examples and Question

We begin with the map $\Phi[a,b,c] : M_3 \to M_3$ defined by

$$\Phi[a,b,c] : x \mapsto \begin{pmatrix} ax_{11}+bx_{22}+cx_{33} & 0 & 0 \\ 0 & ax_{22}+bx_{33}+cx_{11} & 0 \\ 0 & 0 & ax_{33}+bx_{11}+cx_{22} \end{pmatrix} - x$$

for $x = (x_{ij}) \in M_3$, as was studied in [3]. It was shown every $\Phi[a,b,c]$ with the condition

$$1 < a < 3, \qquad 4bc = (3-a)^2$$

gives rise to an element of $\partial\mathbb{D}\cap\operatorname{int}\mathbb{P}_1$, whenever $b \neq c$. We also have a decomposition

$$\Phi[a,b,c] = \frac{a-1}{2}\Phi[3,0,0] + \frac{3-a}{2}\Phi\left[1, \sqrt{\frac{b}{c}}, \sqrt{\frac{c}{b}}\right]$$

into the sum of a completely positive map and a completely copositive map. If we fix b and c, then we see that the family $\{\Phi[a,b,c] : 1 \le a \le 3\}$ is a line segment, and so it suffices to consider the map $\Phi[2,b,c]$. We also see that

$$\Phi[3,0,0] = \phi_{V_1} + \phi_{V_2} + \phi_{V_3}$$

with

$$V_1 = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \quad V_2 = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}, \quad V_3 = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \end{pmatrix},$$

where $\cdot$ denotes 0, and

$$\Phi\left[1, \sqrt{\frac{b}{c}}, \sqrt{\frac{c}{b}}\right] = \phi^{W_1} + \phi^{W_2} + \phi^{W_3}$$

with

$$W_1 = \begin{pmatrix} \cdot & \mu & \cdot \\ -\lambda & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \qquad W_2 = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \mu \\ \cdot & -\lambda & \cdot \end{pmatrix}, \qquad W_3 = \begin{pmatrix} \cdot & \cdot & -\lambda \\ \cdot & \cdot & \cdot \\ \mu & \cdot & \cdot \end{pmatrix},$$

where $\lambda = \left(\frac{b}{c}\right)^{1/4}$ and $\mu = \left(\frac{c}{b}\right)^{1/4}$, and so $\lambda\mu = 1$ and $\lambda \neq 1$.

To find an element of $\Phi[2,b,c]'$, we have to consider the orthogonal complement of the space of 3×3 matrices spanned by $\{V_i, W_i : i = 1,2,3\}$. To do this, we write

$$x = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}, \quad y_1 = \begin{pmatrix} \cdot & \lambda & \cdot \\ \mu & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \quad y_2 = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \lambda \\ \cdot & \mu & \cdot \end{pmatrix}, \quad y_3 = \begin{pmatrix} \cdot & \cdot & \mu \\ \cdot & \cdot & \cdot \\ \lambda & \cdot & \cdot \end{pmatrix},$$

where $\lambda\mu = 1$, $\lambda \neq 1$. Then, it is immediate that

$$\begin{aligned} A &= \widetilde{x}\widetilde{x}^* + \sum_{i=1}^{3} \widetilde{y}_i\widetilde{y}_i^* \\ &= \left(\begin{array}{ccc|ccc|ccc} 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \lambda^2 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \mu^2 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \hline \cdot & 1 & \cdot & \mu^2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \lambda^2 & \cdot & 1 & \cdot \\ \hline \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \lambda^2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \mu^2 & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \end{array}\right), \end{aligned}$$

belongs to the face $\Phi[2, b, c]'$ of $\mathbb{T}$, where

$$\lambda\mu = 1, \qquad \lambda > 0, \qquad \mu > 0, \qquad \lambda \neq 1.$$

This gives us a family of entanglements with positive partial transposes. It has been shown in [12] that these entanglements generate extreme rays in the cone $\mathbb{T}$, and have the Schmidt numbers two by explicit constructions. We note that

$$A' = \text{conv}\,\{\phi_V + \phi^W : V, W \in D\} = \sigma(D, D)$$

where D denotes the orthogonal complement of span $\{x, y_1, y_2, y_3\}$, which is generated by $\{V_1, V_2, V_3, W_1, W_2, W_3\}$.

Problem 1: *Characterize all extreme rays of the cone $\mathbb{T}$ for $m = n = 3$.*

One of the useful methods to construct entanglements with positive partial transposes is to use the notion of unextendible product basis as was considered in [2], [7], [8]. It is easy to see that the four dimensional subspace $D^\perp$ of M_3 has no rank one matrix. On the other hand, the five dimensional subspace D has only six following rank one matrices

$$\begin{pmatrix} 1 & \mu & 0 \\ -\lambda & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & \mu \\ 0 & 0 & 0 \\ -\lambda & 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \mu \\ 0 & -\lambda & -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & \mu & 0 \\ -\lambda & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & \mu \\ 0 & 0 & 0 \\ -\lambda & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & \mu \\ 0 & -\lambda & 1 \end{pmatrix}$$

up to scalar multiplication, which span D. But, no five of them are orthogonal, and so, we see that our example does not come out from unextendible product basis. It is clear that Problem 1 has a close relation with the classification of 4-dimensional subspaces of M_3 due to the rank one matrices in the subspaces. This seems to be the key step for the following question:

Problem 2: *Is every 2-positive linear map between M_3 decomposable? Equivalently, is the inclusion $\mathbb{T} \subset \mathbb{V}_2$ hold for $M_3 \otimes M_3$?*

[Added in the proof (Dec 19, 2003]: It has benn shown in [K.-C. Ha, S.-H. Kye, Entangled states with positive partial transposes arising from indecomposable positive linear maps, II] that every entanglement with positive partial transpose arises in the way described in this survey.

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