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ON UNSTABLE K-GROUPS

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Let X be a finite complex. Consider the based homotopy set

$$U_n(X) = [X, U(n)].$$

Since $U(n)$ is a topological group, $U_n(X)$ becomes a group. By [6], $U_n(X)$ is a nilpotent group.

If n is sufficiently large, $U_n(X)$ merely equals to $\widetilde{K}^1(X)$. In fact if $\dim X < 2n$, $U_n(X)$ is isomorphic to $\widetilde{K}^1(X)$. Therefore we may say that $U_n(X)$ is " the unstable $\widetilde{K}^1$-group of X ".

In this note we consider the case $\dim X = 2n$. Denote the n-th Chern class and Chern character be c_n and ch_n. Put $s_n = n! ch_n$. Since $s_n \in H^{2n}(BU(\infty); \mathbf{Z})$ and is primitive, we can define a homomorphism

$$\theta_X : \widetilde{K}^0(X) \to H^{2n}(X; \mathbf{Z})$$

by $\theta_X(a) = s_n(a)$ where $a \in \widetilde{K}^0(X)$. Our results are the following :

Theorem 1. *If $\dim X \leq 2n$, we have the following central extension :*

$$0 \to \operatorname{Coker} \theta_X \xrightarrow{\iota} U_n(X) \to \widetilde{K}^1(X) \to 0$$

and the order of any element of $\operatorname{Coker}\theta_X$ *divides* $n!$.

Theorem 2. *Let X_1, X_2 be finite complexes of dimensions $2n_1, 2n_2$ respectively. Assume $\widetilde{K}^0(X_1)$ is free, $\widetilde{K}^1(X_1) = 0$ and $H^{2n_1}(X_1; \mathbf{Z}) = H^{2n_2}(X_2; \mathbf{Z}) = \mathbf{Z}$. If* $\operatorname{Coker} \theta_{X_1} = \mathbf{Z}/l_1\mathbf{Z}$, $\operatorname{Coker} \theta_{X_2} = \mathbf{Z}/l_2\mathbf{Z}$, *then* $\operatorname{Coker} \theta_{X_1 \wedge X_2} = \mathbf{Z}/\binom{n_1+n_2}{n_1} l_1 l_2 \mathbf{Z}$.

Example 3.

$$U_n(S^{2n}) = \mathbf{Z}/n!\mathbf{Z} \tag{1}$$

$$U_n(\mathbf{C}P^n) = \{0\} \tag{2}$$

Note that $H^*(U(n); \mathbf{Z}) = \bigwedge(x_1, x_3, \ldots, x_{2n-1})$ where $x_{2j-1} = \sigma(c_j)$.

Theorem 4. *In the same condition as Theorem 1, for any $\tilde{\alpha}, \tilde{\beta} \in U_n(X)$,their commutator $[\tilde{\alpha}, \tilde{\beta}]$ lies in $\iota(\operatorname{Coker} \theta_X)$ and have*

$$[\,\tilde{\alpha}, \tilde{\beta}\,] = \iota\langle u \rangle$$

where

$$u = \sum_{k+l+1=n} \tilde{\alpha}^*(x_{2k+1}) \cup \tilde{\beta}^*(x_{2l+1})$$

in $H^{2n}(X; \mathbf{Z})$ and $\langle u \rangle \in \operatorname{Coker} \theta_X$ represented by $u \in H^{2n}(X; \mathbf{Z})$.

Therefore if $\widetilde{K}^1(X) \neq 0$, then $U_n(X)$ may be non-abelian. In fact we have :

Example 5. *If $S^{2n+1} \to X \to S^{2m+1}$ is a fibration where $0 < n < m$, then $U_{n+m+1}(X)$ has three generators α, β and ϵ and relations are*

$$[\,\alpha, \epsilon\,] = [\,\beta, \epsilon\,] = 0$$
$$(n+m+1)!\epsilon = 0$$
$$[\,\alpha, \beta\,] = n!m!\epsilon$$

For Theorem 4 and Example 5 see [2].

Let G_2 be the compact, simply connected exceptional Lie group of dimension 14. Note that

$$G_2 \simeq S^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14}.$$

Denote $G_2^{(6)}$ by X and $G_2^{(11)}/X$ by Y. Consider the cofiberings

$$X \to G_2^{(11)} \xrightarrow{\pi} Y \xrightarrow{\alpha} \Sigma X \to \Sigma G_2^{(11)} \xrightarrow{\Sigma\pi} \Sigma Y.$$

Using the result of [5], we have (see [3]) :

Theorem 6.

$$U_6(\Sigma G_2^{(11)}) \cong \mathbf{Z}/12\mathbf{Z}$$

In this note we give an elementary proof of Theorem 5. On the other hand $U_6(\Sigma X \vee \Sigma Y) \cong U_6(\Sigma Y) \cong \mathbf{Z}/360\mathbf{Z}$, therefore $\alpha \not\simeq 0$ (stably) by Theorem 2.

1. Proof of Theorem 1 and Theorem 2

Proof of Theorem 1. We denote $U(\infty)/U(n)$ by W_n. From the fibration

$$W_n \xrightarrow{i} BU(n) \xrightarrow{j} BU(\infty)$$

we can deduce the following fibration sequence

$$\Omega U(\infty) \xrightarrow{\Omega p} \Omega W_n \xrightarrow{\Omega i} U(n) \xrightarrow{\Omega j} U(\infty) \xrightarrow{p} W_n.$$

For a CW-complex X, there is an exact sequence of groups

$$[X, \Omega U(\infty)] \xrightarrow{(\Omega p)_*} [X, \Omega W_n] \xrightarrow{(\Omega i)_*} U_n(X) \xrightarrow{(\Omega j)_*} \widetilde{K}^1(X).$$

Since W_n is $2n$-connected, $[X, W_n]$ is trivial. Let x be a generator of $H^{2n+1}(W_n; \mathbf{Z}) \cong \mathbf{Z}$. Since $p^*x = \pm\sigma(c_{n+1})$, we may assume $p^*x = \sigma(c_{n+1})$. Consider $a_{2n} = \sigma(x)$ as a map $\Omega W_n \to K(\mathbf{Z}, 2n)$. Using Whitehead's theorem, we have

$$a_{2n*} : [X, \Omega W_n] \to [X, K(\mathbf{Z}, 2n)] = H^{2n}(X; \mathbf{Z})$$

is bijective. Note that a_{2n} is a loop map and a_{2n*} is a group isomorphism. Consider the Bott equivalence $\beta : BU(\infty) \to \Omega U(\infty)$. Using

$$\beta^* \circ (\Omega p)^*(a_{2n}) = \beta^*(\sigma^2(c_{n+1})) = s_n,$$

we have the following commutator diagram

$$\begin{array}{ccccc} \widetilde{K}^0(X) & & \xrightarrow{\theta_X} & & H^{2n}(X;\mathbf{Z}) \\ \cong\downarrow & & & & \downarrow\cong \\ [X, BU(\infty)] & \xrightarrow[\beta_*]{\cong} & [X, \Omega U(\infty)] & \xrightarrow[(\Omega p)_*]{} & [X, \Omega W_n] \end{array}$$

Denote the commutator map $U(n) \wedge U(n) \to U(n)$ by γ. If $\alpha \in \operatorname{Im}\iota$, then α has a lift $\tilde{\alpha} : X \to \Omega W_n$. For any map $\beta : X \to U(n)$, the commutator $[\,\alpha, \beta\,]$ is given by the composition

$$X \xrightarrow{\triangle} X \wedge X \xrightarrow{\tilde{\alpha}\wedge\beta} \Omega W_n \wedge U(n) \xrightarrow{(\Omega p)\wedge 1} U(n) \wedge U(n) \xrightarrow{\gamma} U(n).$$

Since $\Omega W_n \wedge U(n)$ is $2n$-connected, $[\,\alpha, \beta\,] = 0$. □

Proof of Theorem 2. Since $\widetilde{K}^0(X_1)$ is free and $\widetilde{K}^1(X_1) = 0$, $\widetilde{K}^0(X_1 \wedge X_2) = \widetilde{K}^0(X_1) \otimes \widetilde{K}^0(X_2)$. Let $\theta_j \in \widetilde{K}^0(X_j)$ $(j = 1, 2)$. Since $\dim X_j = 2n_j$,

$$ch_{n_1+n_2}(\theta_1 \otimes \theta_2) = ch_{n_1}(\theta_1) \otimes ch_{n_2}(\theta_2).$$

Therefore

$$\begin{aligned} s_{n_1+n_2}(\theta_1 \otimes \theta_2) &= (n_1 + n_2)!ch_{n_1+n_2}(\theta_1 \otimes \theta_2) \\ &= \binom{n_1+n_2}{n_1} n_1!ch_{n_1}(\theta_1) \otimes n_2!ch_{n_2}(\theta_2) \\ &= \binom{n_1+n_2}{n_1} s_{n_1}(\theta_1) \otimes s_{n_2}(\theta_2). \end{aligned}$$

Therefore we can easily get Theorem 2. The proof of $n!x = 0$ for any $x \in \operatorname{Coker}\theta_X$ is given after the proof of Example 3. □

Proof of Example 3. (1) By induction on n. If $n = 1$ $U_1(S^2) = \pi_2(U(1)) = \{0\}$. If $U_{n-1}(S^{2n-1}) \cong \mathbf{Z}/(n-1)!\mathbf{Z}$, then by Theorem 2,

$$U_n(S^{2n}) \cong U_n(S^2 \wedge S^{2n-2}) \cong \mathbf{Z}/\binom{n}{1}(n-1)!\mathbf{Z} \cong \mathbf{Z}/n!\mathbf{Z}.$$

(2) As is well known $H^*(\mathbf{C}P^n;\mathbf{Z}) \cong \mathbf{Z}[t]/(t^{n+1})$ as an algebra, where $|t| = 2$. Let η be the canonical line bundle, then $ch(\eta - 1) = e^t - 1$. Therefore $s_n(\eta - 1) = n!\left(\dfrac{t^n}{n!}\right) = t^n$ and $\theta_{S^{2n}}(\eta - 1) = t^n$. □

Now we can prove $n!x = 0$ for any $x \in \operatorname{Coker}\theta_X$. Consider the natural projection $\pi : X \to X/X^{(2n-1)} \cong S^{2n} \vee \ldots \vee S^{2n}$ and the following commutative diagram

$$\begin{array}{ccccc} & & \widetilde{K}^0(X/X^{(2n-1)}) & \xrightarrow{\pi^*} & \widetilde{K}^0(X) \\ & & \theta_{X/X^{(2n-1)}}\downarrow & & \downarrow\theta_X \\ \bigoplus H^{2n}(S^{2n};\mathbf{Z}) & \xrightarrow{\cong} & H^{2n}(X/X^{(2n-1)};\mathbf{Z}) & \xrightarrow[\pi^*]{} & H^{2n}(X;\mathbf{Z}). \end{array}$$

Note that π^* is epic on $H^{2n}(\ \ ;\mathbf{Z})$. Therefore we have $n!x = 0$ for any $x \in \operatorname{Coker}\theta_X$ by (1) of Example 3.

2. Proof of Theorem 6

Denote the projection $\Sigma Y \to S^{12}$ by p and inclusions $\Sigma X \to \Sigma G_2^{(11)}$, $G_2^{(11)} \to Spin(7)$ and $Spin(7) \to SU(7)$ by ι, i and j respectively. Note that $\widetilde{K}^0(\Sigma X) \cong \widetilde{K}^0(\Sigma Y) \cong \mathbf{Z}$ and $\widetilde{K}^1(\Sigma X) \cong \widetilde{K}^1(\Sigma Y) = 0$. By the Atiyah-Hirzebruch spectral sequence $p^* : \widetilde{K}^0(S^{12}) \to \widetilde{K}^0(\Sigma Y)$ is monic and $p^*(\widetilde{K}^0(S^{12})) = 2\widetilde{K}^0(\Sigma Y)$. Using the fact $\operatorname{Im}\theta_{S^{12}} = 6!H^{12}(S^{12};\mathbf{Z})$ (see Example 3) and by the commutative diagram

$$\begin{array}{ccc} \widetilde{K}^0(S^{12}) & \xrightarrow{p^*} & \widetilde{K}^0(\Sigma Y) \\ \theta_{S^{12}}\big\downarrow & & \big\downarrow\theta_{\Sigma Y} \\ H^{12}(S^{12};\mathbf{Z}) & \underset{p^*}{\xrightarrow{\cong}} & H^{12}(\Sigma Y;\mathbf{Z}), \end{array}$$

we have $\operatorname{Coker}\theta_{\Sigma Y} \cong \mathbf{Z}/360\mathbf{Z}$. Using $\widetilde{K}^1(\Sigma Y) = 0$ we have :

Lemma 7. $U_6(\Sigma Y) \cong \mathbf{Z}/360\mathbf{Z}$.

Denote generators of $H^k(G;\mathbf{Z}) \cong \mathbf{Z}$ for $G = G_2, Spin(7), SU(7)$ by x_k, y_k, z_k respectively for $k = 3, 11$. Using $Spin(7)/G_2 = S^7$, we have $i^*(y_k) = x_k$ $(k = 3, 11)$. It is well known $j^*(z_k) = 2y_k$ $(k = 3, 11)$. By the Atiyah-Hirzebruch spectral sequence, we have :

Lemma 8. $\iota^* \circ (\Sigma(j \circ i))^* : \widetilde{K}^0(\Sigma SU(7)) \to \widetilde{K}^0(\Sigma X)$ *is epic.*

Consider the inclusion $\epsilon : \Sigma\mathbf{C}P^5 \subset SU(6) \subset SU(7)$. As is well known $\epsilon^* : H^{11}(SU(7);\mathbf{Z}) \to H^{11}(\Sigma\mathbf{C}P^5;\mathbf{Z})$ is epic. Since $H^{11}(SU(7);\mathbf{Z}) \cong H^{11}(\Sigma\mathbf{C}P^5;\mathbf{Z}) \cong \mathbf{Z}$, $\epsilon^* : H^{11}(SU(7);\mathbf{Z}) \to H^{11}(\Sigma\mathbf{C}P^5;\mathbf{Z})$ is an isomorphism. Note that $(\Sigma\epsilon)^* : \widetilde{K}^0(\Sigma SU(7)) \to \widetilde{K}^0(\Sigma^2\mathbf{C}P^5)$ is epic. By Theorem 1 and Example 3, $U_6(\Sigma^2\mathbf{C}P^5) \cong \mathbf{Z}/6\mathbf{Z}$. Therefore $s_6(\widetilde{K}^0(\Sigma SU(7))) = 6H^{12}(\Sigma SU(7);\mathbf{Z})$. Using the exact sequence

$$0 \longrightarrow \widetilde{K}^0(\Sigma Y) \xrightarrow{(\Sigma\pi)^*} \widetilde{K}^0(\Sigma G_2^{(11)}) \xrightarrow{\iota^*} \widetilde{K}^0(\Sigma X) \longrightarrow 0,$$

we have

$$\widetilde{K}^0(\Sigma G_2^{(11)}) = \operatorname{Im}(\Sigma\pi)^* + \operatorname{Im}(\Sigma(j \circ i))^*.$$

Since $(j \circ i)^*(z_{11}) = 2x_{11}$, we have

$$\theta_{\Sigma G_2^{(11)}}(\operatorname{Im}(\Sigma(j \circ i))^*) = 12H^{12}(\Sigma G_2^{(11)};\mathbf{Z}).$$

On the other hand by Lemma 7

$$\theta_{\Sigma G_2^{(11)}}(\operatorname{Im}(\Sigma\pi)^*) = 360H^{12}(\Sigma G_2^{(11)};\mathbf{Z}).$$

Therefore we have $\operatorname{Coker}\theta_{\Sigma G_2^{(11)}} \cong \mathbf{Z}/12\mathbf{Z}$. Since $\widetilde{K}^1(\Sigma G_2^{(11)}) = 0$, we have Theorem 6. Since $\dim \Sigma X = 7 < 12$ we have $U_6(\Sigma X \vee \Sigma Y) \cong U_6(\Sigma Y) \cong \mathbf{Z}/360\mathbf{Z}$.

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