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TWISTED TENSOR PRODUCTS RELATED TO THE COHOMOLOGY OF THE CLASSIFYING SPACES OF LOOP GROUPS

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In the mid-1970's, Kono, Mimura and Shimada ([24],[25],[23]) have determined the mod p cohomology groups of $BPU(3)$ and BF_4 for $p = 3$, and BE_6 and BE_7 for $p = 2$ using the Rothenberg-Steenrod spectral sequence

$$\mathrm{Cotor}_{H^*(G;\mathbb{Z}/p)}(\mathbb{Z}/p, \mathbb{Z}/p) \Longrightarrow H^*(BG; \mathbb{Z}/p).$$

The notion of a twisted tensor product in the sense of Brown [3] plays an important role in their consideration. Roughly speaking, the twisted tensor product associated with the mod p cohomology A of a Lie group is the tensor product of A and a complex $(\overline{X}_A, d)$ equipped with a perturbed differential, and in addition it is an *economical* injective resolutions of $\mathbb{Z}/p$ as a A-comodule. The advantage of the complex $(\overline{X}_A, d)$ is that it is a *manageable* differential graded algebra (DGA) which gives the cotorsion product $\mathrm{Cotor}_A(\mathbb{Z}/p, \mathbb{Z}/p)$, namely, the E_2-term of the spectral sequence. The computation of the mod p cohomology of classifying spaces of other Lie groups due to Mimura and Sambe ([33],[34],[35]) has also told us that a twisted tensor product as an injective resolution is relevant to the study of the cohomology via the spectral sequence.

Let G be a compact, connected simple Lie group and LG denote the loop group which is an infinite dimensional manifold consisting of all C^∞-maps from the circle to G. Our interest here lies in computing the mod p cohomology of the classifying space BLG of the loop group LG. In order to compute those cohomologies, we give a DGA structure to the twisted tensor products by also perturbing the algebra structure of tensor product $(A, 0) \otimes (\overline{X}_A, d)$. More precisely, we have the following theorem.

Theorem 1. *Each twisted tensor product, which is constructed by Kono, Mimura, Sambe and Shimada, associated with mod p cohomology A of a Lie group is that of the differential graded algebras $(A, 0)$ and $(\overline{X}_A, d)$ in the sense of Hess* [15].

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One may perform the computation of the cohomology $H^*(BLG;\mathbb{Z}/p)$ with the aid of the Rothenberg-Steenrod spectral sequence, whose E_2-term is the cotorsion product $\mathrm{Cotor}_{H^*(LG;\mathbb{Z}/p)}(\mathbb{Z}/p,\mathbb{Z}/p)$. As the first step in computing the spectral sequence, we need to construct an economical injective resolution of $\mathbb{Z}/p$ as an $H^*(LG;\mathbb{Z}/p)$-comodule such as the twisted tensor product mentioned above. However, it seems to be hard to carry it out due to infinite number of algebra generators of $H^*(LG;\mathbb{Z}/p)$ and more complicated coalgebra structure $H^*(LG;\mathbb{Z}/p)$ than that of $H^*(G;\mathbb{Z}/p)$. (For the algebra structure of the homology $H_*(LG;\mathbb{Z}/p)$, see [11], [12], [13] and [14].) We then employ other spectral sequences in considering the cohomology $H^*(BLG;\mathbb{Z}/p)$. It is known that there is a homotopy equivalence between BLG and the free loop space $(BG)^{S^1}$ which is the space of all continuous maps from S^1 to BG. Moreover BLG has the homotopy type of $G\times_{\mathrm{ad}} EG$ which is the total space of the associated bundle

$$G\to G\times_{\mathrm{ad}} EG\to BG$$

to the universal principal G-bundle (see [5, Corollary 3.4]). Here the right action of G to itself is the adjoint action

$$\mathrm{ad}: G\times G\to G$$

defined to be $\mathrm{ad}(g,h)=h^{-1}gh$. Observe that the projection of the bundle defines an $H^*(BG;\mathbb{Z}/p)$-algebra structure on $H^*(BLG;\mathbb{Z}/p)$, which is the same as that induced by the evaluation map $LG\to G$ at zero (see [29, (2.1)]).

We thus see that the following three spectral sequences are also applicable in the study of the cohomology $H^*(BLG;\mathbb{Z}/p)$:

HSS: The Hochschild spectral sequence $\{{}_{HH}E_r^{*,*},d_r\}$ converging to $H^*(X^{S^1};\mathbb{Z}/p)$ as an algebra with

$${}_{HH}E_2^{*,*}\cong HH(H^*(X;\mathbb{Z}/p)),$$

where X is a simply connected space and $HH(\ )$ denotes the Hochschild homology functor. Note that $HH(H^*(X;\mathbb{Z}/p))$ is regarded here as a bigraded algebra with the shuffle product;

${}_{\mathrm{B}}$**EMSS**: The bar type Eilenberg-Moore spectral sequence $\{{}_{B}E_r^{*,*},d_r\}$ converging to $H^*(X^{S^1};\mathbb{Z}/p)$ as an algebra with

$${}_{B}E_2^{*,*}\cong \mathrm{Tor}_{H^*(X;\mathbb{Z}/p)\otimes H^*(X;\mathbb{Z}/p)}(H^*(X;\mathbb{Z}/p),H^*(X;\mathbb{Z}/p));$$

${}_{\mathrm{C}}$**EMSS**: The cobar type Eilenberg-Moore spectral sequence $\{{}_{C}E_r^{*,*},d_r\}$ converging to $H^*(G\times_{\mathrm{ad}} EG;\mathbb{Z}/p)$ as an algebra with

$${}_{C}E_2^{*,*}\cong \mathrm{Cotor}_{H^*(G;\mathbb{Z}/p)}(H^*(G;\mathbb{Z}/p),\mathbb{Z}/p),$$

where the $H^*(G;\mathbb{Z}/p)$-comodule structure of $H^*(G;\mathbb{Z}/p)$ is induced by the right adjoint action $\mathrm{ad}: G\times G\to G$.

Observe that the first and second spectral sequences are of the second quadrant and of the third one is the first quadrant.

In order to construct the first spectral sequence HSS, we rely heavily on the isomorphism

$$H^*(X^{S^1};\mathbb{Z}/p) \cong HH(C^*(X;\mathbb{Z}/p))$$

due to Jones [18], where $C^*(X;\mathbb{Z}/p)$ denotes the normalized cochain complex. Recently, Ndombol and Thomas [38] have proved that the isomorphism preserves the algebra structure under an appropriate product on $HH(C^*(X;\mathbb{Z}/p))$. This fact enables us to define an algebra structure on the HSS. On the other hand, the second spectral sequence ${}_{\mathrm{B}}$EMSS is constructed with the isomorphism

$$\mathrm{Tor}_{C^*(X\times X;\mathbb{Z}/p)}(C^*(X;\mathbb{Z}/p), C^*(X;\mathbb{Z}/p)) \cong H^*(X^{S^1};\mathbb{Z}/p)$$

due to Eilenberg and Moore. Therefore we regard these two spectral sequences as essentially different from each other, although the E_2-terms of the spectral sequences are isomorphic as algebras. Observe that

$$HH(A) \cong \mathrm{Tor}_{A\otimes A}(A, A)$$

as an algebra for any commutative algebra A.

As is well known, the cohomology algebra $H^*(BG;\mathbb{Z}/p)$ is an evenly generated polynomial algebra if and only if $H^*(G;\mathbb{Z})$ has no p-torsion. Therefore the following theorem is easily deduced from [21, Theorem 1] due to Kono and Kozima.

Theorem 2. *Suppose that*

$$H^*(BG;\mathbb{Z}/p) \cong \mathbb{Z}/p[y_1, y_2, \ldots, y_l]$$

as an algebra, where the degree of the generator y_j is even for any j. Then,

$$H^*(BLG;\mathbb{Z}/p) \cong \Lambda(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_l) \otimes \mathbb{Z}/p[y_1, y_2, \ldots, y_l]$$

as an $H^*(BG;\mathbb{Z}/p)$-algebra, where $\deg \bar{y}_j = \deg y_j - 1$.

We also determine the algebra structure of $H^*(BLG;\mathbb{Z}/2)$ by using the bar type EMSS and the Steenrod operation on the spectral sequence when $H^*(BG;\mathbb{Z}/2)$ is a polynomial algebra even if $H^*(G;\mathbb{Z})$ has 2-torsion.

Theorem 3. ([29, Theorem 1.6], [22, Theorem 3.1]) *Suppose that*

$$H^*(BG;\mathbb{Z}/2) \cong \mathbb{Z}/2[y_1, y_2, \ldots, y_l]$$

as an algebra in which generators of odd degree are allowed. Then

$$H^*(BLG;\mathbb{Z}/2) \cong \mathbb{Z}/2[\bar{y}_1, \ldots, \bar{y}_l] \otimes \mathbb{Z}/2[y_1, \ldots, y_l]/(\bar{y}_i^2 + \mathcal{D}Sq^{\deg y_i - 1} y_i\,;\, 1 \le i \le l)$$

as an $H^*(BG;\mathbb{Z}/2)$-algebra, where $\deg \bar{y}_i = \deg y_i - 1$ *and*

$$\mathcal{D} : H^*(BG;\mathbb{Z}/2) \longrightarrow H^{*-1}(BLG;\mathbb{Z}/2)$$

is the module derivation of degree -1 defined by $\mathcal{D}y_i = \bar{y}_i$.

By applying Theorem 3, we have

Theorem 4. (i) *As an* $H^*(BSO(n);\mathbb{Z}/2)=\mathbb{Z}/2[w_2,\dots,w_n]$*-algebra,*

$$H^*(BLSO(n);\mathbb{Z}/2)\cong\mathbb{Z}/2[\bar{w}_2,\dots,\bar{w}_n]\otimes$$
$$\mathbb{Z}/2[w_2,\dots,w_n]\Big/\Big(\bar{w}_k^2+\sum_{0\le i\le k-1,i\ne 1}(\bar{w}_{2k-i-1}w_i+w_{2k-i-1}\bar{w}_i)\Big).$$

(ii) *As an* $H^*(BG_2;\mathbb{Z}/2)=\mathbb{Z}/2[v_4,v_6,v_7]$*-algebra,*

$$H^*(BLG_2;\mathbb{Z}/2)\cong\mathbb{Z}/2[x_3,x_5]\otimes\mathbb{Z}/2[v_4,v_6,v_7]\Big/\begin{pmatrix} x_3^4+x_5v_7+v_6x_3^2\\ x_5^2+x_3v_7+v_4x_3^2\end{pmatrix}.$$

(iii) *As an* $H^*(BF_4;\mathbb{Z}/2)=\mathbb{Z}/2[v_4,v_6,v_7,v_{16},v_{24}]$*-algebra,*

$$H^*(BLF_4;\mathbb{Z}/2)\cong\mathbb{Z}/2[x_3,x_5,x_{15},x_{23}]\otimes$$
$$\mathbb{Z}/2[v_4,v_6,v_7,v_{16},v_{24}]\Big/\begin{pmatrix} x_5^2+x_3v_7+x_3^2v_4\\ x_3^4+x_5v_7+x_3^2v_6\\ x_{15}^2+x_3^2v_{24}+x_{23}v_7\\ x_{23}^2+x_3^2v_{16}v_{24}+v_7x_{15}v_{24}+v_7v_{16}x_{23}\end{pmatrix}.$$

Theorem 5. (i) *As an* $H^*(BSpin(7);\mathbb{Z}/2)=\mathbb{Z}/2[y_4,y_6,y_7,u_8]$*-algebra,*

$$H^*(BLSpin(7);\mathbb{Z}/2)\cong\mathbb{Z}/2[x_3,x_5,z_7]\otimes$$
$$\mathbb{Z}/2[y_4,y_6,y_7,u_8]\Big/\begin{pmatrix} x_5^2+x_3y_7+x_3^2y_4,\\ x_3^4+x_5y_7+x_3^2y_6,\\ z_7^2+x_3^2u_8+z_7y_7\end{pmatrix}.$$

(ii) *As an* $H^*(BSpin(8);\mathbb{Z}/2)=Z/2[y_4,y_6,y_7,y_8,u_8]$*-algebra,*

$$H^*(BLSpin(8);\mathbb{Z}/2)\cong\mathbb{Z}/2[x_3,x_5,x_7,z_7]\otimes$$
$$\mathbb{Z}/2[y_4,y_6,y_7,y_8,u_8]\Big/\begin{pmatrix} x_5^2+x_3y_7+x_3^2y_4,\\ x_3^4+x_5y_7+x_3^2y_6,\\ x_7^2+x_3^2y_8+x_7y_7,\\ z_7^2+x_3^2u_8+z_7y_7\end{pmatrix}.$$

(iii) *As an* $H^*(BSpin(9);\mathbb{Z}/2)=\mathbb{Z}/2[y_4,y_6,y_7,y_8,u_{16}]$*-algebra,*

$$H^*(BLSpin(9);\mathbb{Z}/2)\cong\mathbb{Z}/2[x_3,x_5,x_7,z_{15}]\otimes$$
$$\mathbb{Z}/2[y_4,y_6,y_7,y_8,u_{16}]\Big/\begin{pmatrix} x_5^2+x_3y_7+x_3^2y_4,\\ x_3^4+x_5y_7+x_3^2y_6,\\ x_7^2+x_3^2y_8+x_7y_7,\\ z_{15}^2+x_3^2y_8u_{16}+x_7y_7u_{16}+z_{15}y_7y_8\end{pmatrix}.$$

Recall that if G is a simply connected, compact simple classical Lie group, then the cohomology algebra $H^*(BG;\mathbb{Z}/p)$ is a polynomial algebra except for the case where $(G,p)=(Spin(N),2)$ for $N\ge 10$.

When the cohomology $H^*(BG;\mathbb{Z}/p)$ is not a polynomial algebra, we apply the cobar type EMSS for investigating the structure of $H^*(BLG;\mathbb{Z}/p)$. In order to make an explicit calculation of the spectral sequence, as the first step, we need a manageable injective resolution of $\mathbb{Z}/p$ as an $H^*(G;\mathbb{Z}/p)$-comodule with a multiplication. One of the candidates for such a resolution is the twisted tensor product mentioned above. In fact, we can construct a DGA to compute the cotorsion product $\mathrm{Cotor}_{H^*(G;\mathbb{Z}/p)}(H^*(G;\mathbb{Z}/p),\mathbb{Z}/p)$ as an algebra by making use of the twisted tensor product in Theorem 1. It is difficult to determine explicitly the algebra structure of the cotorsion product in general although we use our DGA in the computation. In a particular case, however, the DGA enables us to determine the cotorsion product and to compute the cobar type EMSS converging to $H^*(BLG;\mathbb{Z}/p)$.

Theorem 6. *As an $H^*(BPU(3);\mathbb{Z}/3) = \mathbb{Z}/3[y_2,y_8,y_{12}]\otimes\Lambda(y_3,y_7)/(y_2y_3, y_2y_7, y_2y_8+y_3y_7)$-module,*

$$H^*(BLPU(3);\mathbb{Z}/3) = \mathbb{Z}/3[x_2,y_2,z_6,y_8,z_8,y_{12}]\otimes\Lambda(x_1,y_3,y_7,z_9,z_{11})/I,$$

where I is the ideal generated by elements

$x_2y_2+x_1y_3,$	$y_2y_3,$	$x_2^3,$
$y_2z_6+x_1y_7,$	$x_1z_8,$	y_2y_7
$z_6y_3-x_2y_7-x_1y_8,$	$x_1z_9-y_2z_8,$	$x_2z_8,$
$y_2y_8+y_3y_7,$	$z_8y_3-x_1x_2^2z_6,$	$x_1z_{11},$
$x_2z_9,$	$z_9y_3-x_1x_2^2y_7,$	$x_1z_6^2+x_2z_{11},$
$y_2z_{11},$	$z_{11}y_3+x_1z_6y_7,$	$z_6z_8,$
$z_6z_9+z_8y_7,$	$z_8y_7-x_2^2z_{11},$	$x_2^2z_6^2-z_8y_8,$
$z_9y_7,$	$z_8^2,$	$x_2^2z_6y_7+z_9y_8,$
$z_6z_{11}+x_1x_2^2y_{12},$	$z_8z_9,$	$z_6^3,$
$z_{11}y_7+x_1x_2y_3y_{12},$	$-z_6^2y_7+z_{11}y_8+x_2^2y_3y_{12},$	$z_8z_{11},$
$z_9z_{11},$		

Theorem 4, (i) of Theorem 5 and Theorem 6 are known also to A. Kono ([20]).

Theorem 7. *As an $H^*(BSpin(10);\mathbb{Z}/2)=\mathbb{Z}/2[y_4,y_6,y_7,y_8,y_{10},y_{32}]/(y_7y_{10})$-module,*

$$\begin{aligned} H^*(BLSpin(10);\mathbb{Z}/2) &\cong \mathrm{Cotor}_{H^*(Spin(10);\mathbb{Z}/2)}(H^*(Spin(10);\mathbb{Z}/2),\mathbb{Z}/2)\\ &\cong \mathbb{Z}/2[x_3,y_4,y_6,y_7,y_8,y_{10},y_{32}]\\ &\qquad\otimes\Lambda(x_5,x_7,x_9,z_{30},z_{31},w_{31})/I, \end{aligned}$$

where I is the ideal generated by elements

$x_3^4,$	$y_7y_{10},$	$w_{31}y_{10},$	$z_{31}y_7,$	$x_3^2z_{30},$	$x_3^2z_{31},$	$x_9y_7+x_3^2y_{10},$
$x_9z_{30},$	$x_9w_{31},$	$z_{30}z_{31},$	$z_{30}w_{31},$	$z_{31}w_{31},$	$x_3^2w_{31}+z_{30}y_7,$	$x_9z_{31}+z_{30}y_{10}.$

As mentioned above, cohomology algebra $H^*(BSpin(N);\mathbb{Z}/2)$ is not a polynomial algebra for $N \geq 10$. Therefore, it is worthwhile to compute as *the first example* to which Theorem 3 is not applicable.

We denote by $\{{}_{HH}E_r^{*,*}, d_r\}$ and $\{{}_{C}E_r^{*,*}, d_r\}$ the HSS and the cobar type EMSS respectively converging to the same cohomology algebra $H^*(BLSpin(10);\mathbb{Z}/2)$.

Let us here describe the outline of the proof of Theorem 7, because the proof itself is of our interest. What we pay attention to in the proof is to exchange information on the triviality of spectral sequences between $\{{}_{HH}E_r^{*,*}, d_r\}$ and $\{{}_{C}E_r^{*,*}, d_r\}$. The procedure is stated as follows. Let A be the cohomology Hopf algebra $H^*(Spin(10);\mathbb{Z}/2)$. We first calculate exactly the cotorsion product $\mathrm{Cotor}_A(A, \mathbb{Z}/2)$ by using the twisted tensor product associated with A, which we shall construct later in this paper. Consequently, we see that the E_2-term ${}_{C}E_2^{*,*}$ is generated by elements with total degree less than or equal to 32. A partial calculation of the Hochschild homology $HH(H^*(BSpin(10);\mathbb{Z}/2))$, which is the E_2-term of the spectral sequence $\{{}_{HH}E_r^{*,*}, d_r\}$, allows us to compare ${}_{HH}E_2^{*,*}$ with ${}_{C}E_2^{*,*}$ as a vector space up to total degrees 45. In consequence, we have the following key lemma to prove Theorem 7.

Lemma 8. *For any integer $j \leq 45$, the HSS*

$$E_2 = HH(H^*(BSpin(10);\mathbb{Z}/2)) \Longrightarrow H^*(BLSpin(10);\mathbb{Z}/2)$$

collapses at the E_2-term for total degree below j if and only if so does the cobar type EMSS

$$E_2 = \mathrm{Cotor}_{H^*(Spin(10);\mathbb{Z}/2)}(H^*(Spin10;\mathbb{Z}/2), \mathbb{Z}/2) \Longrightarrow H^*(BLSpin(10);\mathbb{Z}/2).$$

In particular, if the HSS collapses at the E_2-term for total degree ≤ 32, then the cobar type EMSS collapses at the E_2-term.

For dimensional reasons, it follows that the HSS $\{{}_{HH}E_r^{*,*},\ d_r\}$ collapses at the E_2-term for total degree ≤ 29. By virtue of Lemma 8, we see that so does the cobar type EMSS $\{{}_{C}E_r^{*,*}, d_r\}$. From the knowledge of the Steenrod operation on $\{{}_{C}E_r^{*,*}, d_r\}$ and the algebra structure of $\mathrm{Cotor}_A(A, \mathbb{Z}/2)$, and further, by comparing the E_1-term with that of the cobar type EMSS converging to $H^*(BLE_6;\mathbb{Z}/2)$, we can deduce that an algebra generator of ${}_{C}E_2^{0,30}$ is a permanent cycle. It follows from Lemma 8 that the HSS $\{{}_{HH}E_r^{*,*}, d_r\}$ also collapses at the E_2-term for total degree ≤ 30. We find just two algebra generators of ${}_{HH}E_2^{*,*}$ with total degree 31. One is in ${}_{HH}E_2^{-1,32}$ and the other is in ${}_{HH}E_2^{-3,34}$. It is immediate to see that the generator in ${}_{HH}E_2^{-1,32}$ is a permanent cycle. If the generator in ${}_{HH}E_2^{-3,34}$, say z, is also a permanent cycle, then we see that the HSS collapses at the E_2-term for total degree ≤ 32, because the only generator with total degree 32 is in ${}_{HH}E_2^{0,32}$. Lemma 8 implies that the cobar EMSS collapses at the E_2-term. We thus obtain Theorem 7. In order to prove that the element z in ${}_{HH}E_2^{-3,34}$ is a permanent cycle, we take the TV-model

$$\alpha : (TV_{BSpin(10)}, d) \xrightarrow{\simeq} C^*(BSpin(10); \mathbb{Z}/2)$$

which is a quasi-isomorphism; that is, the map induces an isomorphism on the cohomology. We compare the HSS $\{{}_{HH}\tilde{E}_r^{*,*}, \tilde{d}_r\}$ converging to $HH(TV_{BSpin(10)})$ with $\{{}_{HH}E_r^{*,*}, d_r\}$ converging

$$HH(C^*(BSpin(10); \mathbb{Z}/2)) \cong H^*(BLSpin(10); \mathbb{Z}/2)$$

by the isomorphism of spectral sequences induced by α. Observe that the differential of $\{{}_{HH}\tilde{E}_r^{*,*}, \tilde{d}_r\}$ is dominated by the differential d of the TV-model for $BSpin(10)$ (see [26, Lemma 2.1]). By analyzing the TV-model, we can conclude that the algebra generator $\tilde{z}$ in ${}_{HH}\tilde{E}_2^{-3,34}$, which corresponds to the element z, is a permanent cycle and hence so is z.

As for the TV-model (TV_X, d) for a simply connected space X, it follows from [10] that the vector space V_X is isomorphic to the suspension of cohomology $H^*(\Omega X; \mathbb{Z}/p)$ and the quadratic part of the differential d can be identified with the coproduct of $H^*(\Omega X; \mathbb{Z}/p)$. In general, it is by no means easy to determine the *higher part* of the differential d of the TV-model. Even in the TV-model $(TV_{BSpin(10)}, d)$, we can not deduce an exact form of the differentials $\tilde{d}_r$ of $\{{}_{HH}\tilde{E}_r, \tilde{d}_r\}$. To this end, such incomplete information on the TV-model does not work well in determining differentials on ${}_{HH}\tilde{E}_r^{*,*}$ with total degree 30, although, as mentioned above, it is possible to conclude that the differential $\tilde{d}_r$ on ${}_{HH}\tilde{E}_r^{*,*}$ with total degree 31 is trivial. Fortunately, we obtain an explicit form of an important differential of $(TV_{BSpin(10)}, d)$ from the triviality of the cobar type EMSS $\{{}_{C}E_r^{*,*}, d_r\}$ (see Theorem 7). This fact leads us to an application of the TV-model.

Let $\mathbb{T}$ be the circle group and $\alpha : \mathbb{T} \times BLG \to BLG$ the circle action on BLG which is induced by the $\mathbb{T}$-action on the loop group LG defined by $(t\gamma)(s) = \gamma(ts)$ for $t \in \mathbb{T}$ and $\gamma \in LG$. We define the map

$$\lambda : H^*(BLG; \mathbb{Z}/p) \longrightarrow H^*(BLG; \mathbb{Z}/p)$$

of degree -1 by $\int_{S^1} \circ \alpha^*$, where $\int_{S^1}$ is the integration along the circle. Observe that λ is a derivation. A Hochschild homological interpretation of the map λ due to Jones [18] enables us to deduce the following proposition.

Proposition 9. *There exists a decreasing filtration*

$$F^* = \{F^i H^*(BLG; \mathbb{Z}/p)\,;\, i \leq 0\}$$

of $H^*(BLG; \mathbb{Z}/p)$ *such that the map* λ *decreases the filtration degree by* 1:

$$\lambda : F^i H^{i+j}(BLG; \mathbb{Z}/p) \longrightarrow F^{i-1} H^{i-1+j}(BLG; \mathbb{Z}/p).$$

From the knowledge about the differential of the TV-model for $BSpin(10)$, we have a result concerning the map λ in the case $G = Spin(10)$.

Theorem 10. *There exist algebra generators* $w_i, \bar{w}_i$ $(i = 4, 6, 7, 8, 10, 32)$, ξ_{30} *and* ξ_{31}, *which are in the filtrations* F^0, F^{-1}, F^{-3} *and* F^{-4} *of* $H^*(BLSpin(10);\mathbb{Z}/2)$, *respectively, such that* $\lambda(w_i) = \bar{w}_i$, $\lambda(\bar{w}_i) = \lambda(\xi_{31}) = 0$ *and* $\lambda(\xi_{30})$ *is in the filtration* F^{-3}.

As will be seen, our application of the notions of a twisted tensor product and of TV-models is far from a general theory. However, the novelty here is not simply in the explicit calculation of the cohomology $H^*(BLSpin(10);\mathbb{Z}/2)$ but also in the new usage of TV-models combining with spectral sequences. The authors believe firmly that the manner of such explicit calculation becomes to be a seed to develop a theory of algebraic models for spaces.

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