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ON THE ALEXANDER POLYNOMIALS OF PERIODIC LINKS AND RELATED TOPICS

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ABSTRACT. In this article we give a survey of the results that are known on the Alexander polynomials of periodic knots and links in S^3 and some related topics.

1. INTRODUCTION

A link ℓ in S^3 is said to have *period* $n (n \geq 2)$ if there is an n-periodic homeomorphism ϕ from S^3 onto itself such that ℓ is invariant under ϕ and the fixed point set f of the $\mathbb{Z}_n$-action induced by ϕ is homeomorphic to a 1-sphere in S^3 disjoint from ℓ. By the positive solution of the Smith Conjecture [28], f is unknotted and so the homeomorphism ϕ is conjugate to one point compactification of the standard $\frac{2\pi}{n}$-rotation about the z-axis in $\mathbb{R}^3$. Hence the quotient map $\pi : S^3 \to S^3/\mathbb{Z}_n$ is an n-fold cyclic cover branched along $\pi(f) = f_*$, and $\ell_* = \pi(\ell)$ is also a link in the orbit space $S^3/\mathbb{Z}_n \cong S^3$, which is called the *factor link* of ℓ.

A natural question is how to determine whether a link is periodic with a given period. In 1962, Fox[11] conjectured that a non trivial knot has only finitely many periods. This conjecture was first proved by Flapan[9] in 1983 and Hillman[16] extended her argument to apply to links in 1984. But in neither case was any kind of applicable bound established for the periods of a given knot or link. In 1984, an explicit upper bound for possible periods of a knot in terms of its genus was given by Edmonds[8] and in 1994, Naik[34] sharpened this upper bound by using the Murasugi conditions[7, 32] and his new results on the homology groups of the finite cyclic covers of S^3 branched along periodic knots. In general it is hard to find the periods of a given knot or link. Much of study on periodic knots and links concerns the criteria to determine the possible periods for a given knot or link and the problem: Which invariants or properties of periodic knots and links are determined by those of their factor knots or links? Up to now many available techniques have been developed for determining the possible periods of knots and links. The first significant results were those of Trotter[48] on the periods of torus knots(cf.[6]) and some simple knots, obtained by analyzing possible actions on the

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fundamental group of the knot. Murasugi's study[32] for the Alexander polynomials of periodic knots proved especially powerful and further work on the Alexander polynomial and some related invariants were done by many authors and these are the subject of this survey article. Some results on the Jones polynomial and its generalizations[33, 38, 39, 47, 51] and hyperbolic structures on knot complements[1] have been applied to the study of periodic knots and links as well.

The purpose of this paper is to give a survey of the results that are known on the Alexander polynomials of periodic knots and links in S^3 and related topics; the genera of periodic knots, the signatures of periodic links, and the homology modules of the finite cyclic covers of S^3 branched along periodic knots.

The paper is organized as follows. In Section 2, we discuss the Alexander polynomials of periodic knots and links. In Section 3, we discuss upper bounds of the possible periods of a given knot in terms of the genus and the Alexander polynomial of the knot. In Section 4, we discuss some recent results on the signatures of periodic links. Finally, in Section 5, we consider the finite cyclic covers of S^3 branched along periodic knots. The proofs are referred to the original literature for details.

2. The Alexander polynomials of periodic links

Let $\ell = k_1 \cup \cdots \cup k_\mu$ be an oriented link in S^3 of μ components, let E be the exterior of ℓ, and let $\pi_1(E)$ be the link group of ℓ. Let t_i be the homology class in $H_1(E)$ represented by a meridian of $k_i (1 \leq i \leq \mu)$. Then $H_1(E)$ is a free abelian group of rank μ generated by $t_1, \cdots, t_\mu$. Let $\gamma : \pi_1(E) \to H_1(E)$ be the Hurewicz epimorphism and let E_γ be the universal abelian covering space of E corresponding to the kernel of γ. Then $H_1(E)$ acts on E_γ as the covering transformation group and so $H_1(E_\gamma)$ can be regarded as a module over the integral group ring $\mathbb{Z}H_1(E)$ of $H_1(E)$. By regarding $H_1(E)$ as the multiplicative free abelian group F_μ with basis $t_1, \cdots, t_\mu$, we can identify $\mathbb{Z}H_1(E)$ with the Laurent polynomial ring $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \cdots, t_\mu^{\pm 1}]$ in the variables $t_1, \cdots, t_\mu$, so that we can regard $H_1(E_\gamma)$ as a Λ-module. The 0-th characteristic polynomial of $H_1(E_\gamma)$, i.e., the greatest common divisor of the elements of the 0-th elementary ideal of $H_1(E_\gamma)$, is called the *Alexander polynomial of ℓ on μ variables*, and denoted by $\Delta_\ell(t_1, \cdots, t_\mu)$.

Now let $\nu : H_1(E) \to F_r$ be an epimorphism from $H_1(E)$ to the free abelian group F_r of rank r with basis $t_1, \cdots, t_r$ and let E_ν be the covering space over E corresponding to the kernel of the composite homomorphism $\nu \circ \gamma : \pi_1(E) \to F_r$. Then $H_1(E_\nu)$ can be regarded as a $\mathbb{Z}F_r$-module. The *reduced Alexander polynomial of ℓ on r variables associated to ν* is defined to be the 0-th characteristic polynomial of the $\mathbb{Z}F_r$-module $H_1(E_\nu)$ and denoted by $\tilde{\Delta}_\ell(t_1, \cdots, t_r)$. If ℓ is a knot, we have $\tilde{\Delta}_\ell(t) \doteq \Delta_\ell(t)$. For $\mu \geq 2$, the relationship between the Alexander polynomial $\Delta_\ell(t_1, \cdots, t_\mu)$ and the reduced one $\tilde{\Delta}_\ell(t_1, \cdots, t_r)$ is as follow[22, Proposition 7.3.10]:

$$(2.1) \qquad \begin{cases} \tilde{\Delta}_\ell(t_1) \doteq (t_1 - 1)\Delta_\ell(\nu(t_1), \cdots, \nu(t_\mu)) & \text{if } r = 1, \\ \tilde{\Delta}_\ell(t_1, \cdots, t_r) \doteq \Delta_\ell(\nu(t_1), \cdots, \nu(t_\mu)) & \text{if } r \geq 2. \end{cases}$$

The (reduced) Alexander polynomials of links can be calculated by using various methods[2, 4, 10, 22, 40, 42]. On the other hand, it is well known that any Alexander polynomial $\Delta_k(t)$ of a knot k in S^3 satisfies the following two conditions:

$$\text{(2.2)} \qquad \text{(i) } \Delta(1) = \pm 1, \text{(ii) } \Delta(t) = t^{\deg \Delta}\Delta(t^{-1}).$$

Conversely, for any given polynomial $\Delta \in \mathbb{Z}[t]$ satisfying (2.2), there exists a knot k with $\Delta_k(t) = \Delta$[26, 42, 43]. By definition a polynomial $\Delta \in \mathbb{Z}[t]$ is called a *knot polynomial* if it satisfies the two conditions of (2.2).

Now let $\ell = k_1 \cup \cdots \cup k_\mu$ be an oriented link in S^3 of μ components and let f be the oriented trivial knot such that $\ell \cap f = \emptyset$. For any integer $n \geq 2$, let $\pi : S^3 \to S^3$ be the n-fold cyclic cover branched along f. We denote the preimage $\pi^{-1}(\ell)$ and $\pi^{-1}(k_i)$ by $\ell^{(n)}$ and $k_i^{(n)}$, respectively. Then $k_i^{(n)} = k_{i1} \cup \cdots \cup k_{i\nu_i}$ is a link of ν_i components, where ν_i is the greatest common divisor of n and $\lambda_i = Lk(k_i, f)$, the *linking number* of k_i and f. We give an orientation to $k_i^{(n)}$ inherited from k_i. Then $\ell^{(n)} = k_1^{(n)} \cup \cdots \cup k_\mu^{(n)} = k_{11} \cup \cdots \cup k_{1\nu_1} \cup \cdots \cup k_{\mu 1} \cup \cdots \cup k_{\mu\nu_\mu}$ is an oriented n-periodic link in S^3 with ℓ as its factor link. Throughout this paper we call such an oriented link $\ell^{(n)}$ the *n-periodic covering link over $\ell_1 = \ell \cup f$*. Notice that every link in S^3 with cyclic period arises in this manner.

In 1971, Murasugi[32] gave a relationship among the reduced Alexander polynomial $\tilde{\Delta}_{\ell^{(n)}}(t)$ of the periodic covering link $\ell^{(n)}$ and the Alexander polynomials $\Delta_k(t), \Delta_{k\cup f}(t_1, t_2)$ for the case that $\ell_1 = k \cup f$ is a link of two components in a homology 3-sphere $\mathcal{M}$. Here we shall state the results for the special case $\mathcal{M} = S^3$:

Theorem 2.1. Let $\ell_1 = k \cup f$, where f is unknotted, and let $\lambda = Lk(\ell, f) \neq 0$. Let $\ell^{(n)}$ be the oriented n-periodic covering link in S^3 over ℓ_1. Then

$$\text{(2.3)} \qquad \tilde{\Delta}_{\ell^{(n)}}(t) \doteq \Delta_k(t) \prod_{i=1}^{n-1} \Delta_{k\cup f}(t, \omega^i),$$

where t in $\tilde{\Delta}_{\ell^{(n)}}(t)$ corresponds to meridians of all components of $\pi^{-1}(k)$ and ω denotes a primitive n-th root of unity.

If $n = p^r m, p$ a prime, $(p, m) = 1, r > 0$, then

$$\delta_\lambda(t)\tilde{\Delta}_{\ell^{(n)}}(t) \equiv [\prod_{i=0}^{m-1} \Delta_{k\cup f}(t, \eta^i)]^{p^r} \pmod p,$$

where $\delta_\lambda(t) = (1 - t^\lambda)/(1 - t)$ and η denotes a primitive m-th root of unity. In particular, if $m = 1$, then

$$\tilde{\Delta}_{\ell^{(n)}}(t) \equiv \Delta_k(t)^{p^r} \delta_\lambda(t)^{p^r - 1} \pmod p.$$

Theorem 2.2. Let $\ell_1 = k \cup f$ be a two component link in S^3, where f is unknotted and $Lk(k, f) = \lambda$. Let $k^{(n)}$ be the n-periodic covering knot over $\ell_1 = k \cup f$, where $n = p^r (r \geq 1)$ and p is a prime with $(\lambda, p) = 1$. Then

$$\Delta_{k^{(n)}}(t) \equiv (1 + t + \cdots + t^{\lambda - 1})^{n-1} \Delta_k(t)^n \pmod p.$$

Corollary 2.3. Suppose that k is a periodic knot of a prime power period $n = p^r$ in S^3. Then the Alexander polynomial $\Delta_k(t)$ of k must satisfy the following:

$$\Delta_k(t) \equiv (1 + t + \cdots + t^{\lambda-1})^{n-1}\Delta(t)^n \pmod p \tag{2.4}$$

for some positive integer $\lambda, (\lambda, p) = 1$, and a certain knot polynomial $\Delta(t)$.

Corollary 2.4. Suppose that k is a periodic knot of a prime power period $n = p^r$ in S^3. If $\Delta_k(t)$ is not a product of other knot polynomials in $\mathbb{Z}(t)$, then for some positive integer $\lambda, (\lambda, p) = 1$,

$$\Delta_k(t) \equiv (1 + t + \cdots + t^{\lambda-1})^{n-1} \pmod p,$$

and hence, for any integer s,

$$\Delta_k(s) \equiv 0 \text{ or } \pm 1 \pmod p.$$

Corollary 2.3 and 2.4 give us useful criteria to detect periodicity of a given knot. Murasugi's results have been extended to the more general case by Hilman[15, 17], Sakuma[41], Turaev[50], and Lee[24]. In 1994, Miyazawa[29] gave a similar relationship among the Conway polynomials of ℓ, ℓ_1 and $\ell^{(n)}$ for periodic links.

Theorem 2.5 ([41])**.** Let $\ell_1 = \ell \cup f = k_1 \cup \cdots \cup k_\mu \cup f \subset S^3$ and let $\ell^{(n)}$ be the n-periodic covering link over $\ell_1 = \ell \cup f$. Then

$$\tilde{\Delta}_{\ell^{(n)}}(t_1, \cdots, t_\mu) \doteq \Delta_\ell(t_1, \cdots, t_\mu) \prod_{j=1}^{n-1} \Delta_{\ell\cup f}(t_1, \cdots, t_\mu, \omega),$$

where $t_i (1 \leq i \leq \mu)$ corresponds to meridians of all components of $\pi^{-1}(k_i) = k_{i1} \cup \cdots \cup k_{i\nu_i}$.

Theorem 2.6 ([24])**.** Let ℓ be an oriented link in S^3 of μ components, let $\ell_1 = \ell \cup f$, where f is unknotted, and let $\lambda = Lk(\ell, f)$. Let $\ell^{(n)}$ be the oriented n-periodic covering link in S^3 over ℓ_1 of period $n = p^r (r \geq 1)$, where p is an odd prime. Then the reduced Alexander polynomials $\tilde{\Delta}_{\ell^{(n)}}(t)$ and $\tilde{\Delta}_\ell(t)$, where a meridian of each component of $\ell^{(n)}$ and ℓ corresponds to t, satisfy the congruence:

$$\tilde{\Delta}_{\ell^{(n)}}(t) \equiv (1 + t + \cdots + t^{\lambda-1})^{n-1}\tilde{\Delta}_\ell(t)^n \pmod p. \tag{2.5}$$

In 1991, Davis and Livingston[7] tried to give a characterization of the Alexander polynomials of periodic knots. To do this end they rephrased the formula (2.3) as follows:

Murasugi Conditions. Let Δ be a knot polynomial, n a positive integer, and $G =< g >$, the cyclic group of order n generated by g. There is a knot polynomial Δ_*, a polynomial $\Delta_G(t, g) \in \mathbb{Z}[G][t, t^{-1}]$, and a positive integer λ with $(\lambda, n) = 1$ such that

(i) Δ_* divides Δ,
(ii) $\Delta/\Delta_* = \prod_{i=1}^{n-1} \Delta_G(t, \omega^i)$,
(iii) $\Delta_G(1, g) = \delta_\lambda(g),\ \Delta_G(t, 1) = \delta_\lambda(t)\Delta_*(t)$,
(iv) $\Delta_G(t^{-1}, g^{-1}) = t^a g^b \Delta_G(t, g)$ for some integer a, b,

where $\delta_\lambda(t) = (1-t^\lambda)/(1-t)$ and ω is a primitive n-th root of unity.

From the Torres condition[45] and the formula (2.3), it is not difficult to see that if Δ is the Alexander polynomial of an n-periodic knot, then the pair (Δ, n) satisfies the Murasugi conditions. Here are the two natural questions on Murasugi conditions: (1) Given a knot polynomial Δ and n a positive integer, when are the Murasugi conditions satisfied? (2) For a given pair (Δ, n), if it satisfies the Murasugi conditions, is there a knot k of period n such that the Alexander polynomial $\Delta_k(t)$ of k is equal to Δ? To answer the question (1) it is necessary to find unknown polynomials $\Delta_* \in \mathbb{Z}[t]$ and $\Delta_G(t,g)$ in $\mathbb{Z}[G][t,t^{-1}]$ satisfying the conditions. But it may be a somewhat difficult algebraic problem. In the case that n is a prime p, Davis and Livingston[7] gave more easily applicable criteria, Modified Murasugi Conditions on (Δ, p), and some practical examples. For the question (2), they presented the following conjecture and obtained some partial answers:

Conjecture. If (Δ, n) satisfies the Murasugi Conditions, then Δ is the Alexander polynomial of a knot of period n.

Theorem 2.7 ([7])**.** If (Δ, n) satisfies the Murasugi Conditions with $\lambda = 1$, then Δ is the Alexander polynomial of a knot k of period n with $\Delta_{k_*}(t) = \Delta_*$.

Corollary 2.8. A knot polynomial Δ which is congruent to 1 modulo n is the Alexander polynomial of a knot of period n.

For a given pair (Δ, n) satisfying the Murasugi Conditions, the construction of an n-periodic knot k with $\Delta_k(t) = \Delta$ is closely related to the classification problem of the Alexander polynomials of two components links. More precisely, in 1953, Torres[45] discovered a formula relating the Alexander polynomial of a two component link to the polynomials of its component knots. Since the possible Alexander polynomials of knots are known[43], there arose the question of whether Torres conditions are sufficient for a two variable polynomial $\Delta(s,t) \in \mathbb{Z}[s,t]$ to be the Alexander polynomial of a two component link. Now it was shown that Torres conditions are insufficient to characterize the Alexander polynomials of two components links $\ell = k_1 \cup k_2$ with the linking number $Lk(k_1, k_2) \geq 3$[18, 37]. While, the Torres conditions are sufficient to characterize the Alexander polynomials of two component links with the linking number 0 and 1[27]. So only the case of linking number 2 is left unresolved. This leads that the above Davis and Livingston's conjecture is answered negatively for the case of linking number $\lambda \geq 3$ and consequently the only case of $\lambda = 2$ is still open.

The remainder of this section will be devoted to discuss the works on the splitting field over $\mathbb{Q}$ of the Alexander polynomial $\Delta_k(t)$ of a periodic knot k. In 1961, Trotter[48] showed that if the commutator subgroup of the group of a knot k with prime power period $n = p^r$ is free and if $\Delta_k(t)$ has no repeated roots, then the n-th roots of unity lie in $\mathrm{Split}(\Delta_k/\mathbb{Q})$, where $\mathrm{Split}(\Delta_k/\mathbb{Q})$ is the splitting field of $\Delta_k(t)$ over $\mathbb{Q}$. In 1978, Burde[5] weakened Trotter's hypotheses to requiring that $\Delta_k(t) \not\equiv 1 \pmod{p}$ and the second Alexander polynomial $\Delta_{2,k}(t)$[10] is trivial and this was extended to all knots by Hillman[19] as a condition involving the higher Alexander polynomials: if $\Delta_k(t) \not\equiv 1 \pmod p$ but $\Delta_{m+1,k}(t) = 1$, then ζ has degree at most m over $\mathrm{Split}(\Delta_k/\mathbb{Q})$, where ζ is a primitive n-th root of unity. The Burde-Trotter and Murasugi conditions have been extended to knots in homology 3-spheres[15]. More recently, Hillman[20] showed that if $\Delta_k(t)$ is the Alexander polynomial of a knot k with prime power period $n = p^r$, then the Burde-Trotter condition implies that $[\mathbb{Q}(\zeta) : \mathbb{Q}(\zeta) \cap \mathrm{Split}(\Delta_k/\mathbb{Q})] \leq m$, where m is the maximal multiplicity of irreducible factors of $\Delta_k(t)$. He also proved the following:

Theorem 2.9. Let Δ be a polynomial in $\mathbb{Z}[t]$ which satisfies the Murasugi conditions for some prime power $n = p^r$. Suppose that Δ has irreducible factorization $\Delta = \prod \delta_i^{e_i}$. Then either $\Delta \equiv 1 \pmod p$ or

$$[\mathbb{Q}(\zeta) : \mathbb{Q}(\zeta) \cap \mathrm{Split}(\Delta/\mathbb{Q})] \leq m = \max\{e_i\}.$$

3. A bound of the periods of a knot

The *genus*, $g(\ell)$, of a link ℓ in S^3 is defined to be the minimum of the genera of Seifert surfaces bounded by ℓ. By means of the theory of least area surfaces and the Riemann-Hurwitz formula for branched coverings of surfaces, Edmonds[8] gave an upper bound of possible periods of a given knot in terms of its genus:

Theorem 3.1. If k is an n-periodic knot with genus g, then k has a genus g Seifert surface which is invariant under the action of $\mathbb{Z}_n$.

Corollary 3.2. If k is an n-periodic knot with genus g, then $n \leq 2g + 1$.

Proof. Let S be a genus g Seifert surface of k which is invariant under the action of $\mathbb{Z}_n$. Then the quotient map $S \to S/\mathbb{Z}_n$ is an n-fold cyclic branched covering with m-branch points, where m is the number of points of intersection of the fixed point set f of the periodic $\mathbb{Z}_n$-action and the equivariant Seifert surface S. By the Riemann-Hurwitz formular, $\chi(S) = n\chi(S/\mathbb{Z}_n) - m(n-1)$. Set $\chi(S) = 1 - 2g$ and $\chi(S/\mathbb{Z}_n) = 1 - 2\bar{g}$. This yields that $n(2\bar{g} + m - 1) = 2g + m - 1$. This implies that $n \leq 2g + 1$. □

Remark 3.3. The proof of Corollary 3.2 shows that one can actually has $n \leq g$ with just two exceptions:

(i) $\bar{g} = 0, m = 2$, and $n = 2g + 1$,
(ii) $\bar{g} = 0, m = 3$, and $n = g + 1$.

The only ways one can have $n = g$ are

(i) $\bar{g} = 0, m = 5$, and $n = g = 2$,
(ii) $\bar{g} = 0, m = 4$, and $n = g = 3$,
(ii) $\bar{g} = 1, m = 1$, and $n = g$.

Corollary 3.4. If k is a nontrivial n-periodic knot with genus g, trivial Alexander polynomial $\Delta_k(t) = 1$, then $n \leq g + 1$.

The following Theorem is an immediate consequence of the argument in the proof of Corollary 3.2, which is known as *Riemann-Hurwitz Formula* for periodic knots:

Theorem 3.5 ([34])**.** Let k be a knot of period n with genus g and let g_* be the genus of the factor knot k_*. Let m be the number of points of intersection of the fixed point set f of the periodic $\mathbb{Z}_n$-action and an equivariant Seifert surface S of k with genus g. Then

$$g = n\bar{g} + \frac{(n-1)(m-1)}{2},$$

where $\bar{g}$ denotes the genus of the surface $S/\mathbb{Z}_n$. In particular, $g \geq ng_*$.

By combining Corollary 3.2, Remark 3.3, and Theorem 2.2, Naik[34] observed the following results, which give us highly efficient criteria in determining the possible periods higher than the genus of a given knot:

Theorem 3.6. Let k be a knot of period n with genus g. If $n = p^r$, p a prime, and $n > g$. Then $g_* = 0$ and exactly one of the followings holds:

(i) $n = g+1, \lambda = 1$, and $\Delta_k(t) \equiv 1 \pmod p$.
(ii) $n = g+1, \lambda = 3, \deg\Delta_k(t) = 2(n-1)$, and $\Delta_k(t) \equiv (1+t+t^2)^{n-1} \pmod p$.
(iii) $n = 2g+1, \lambda = 2, \deg\Delta_k(t) = n-1$, and $\Delta_k(t) \equiv (1+t)^{n-1} \pmod p$.

Corollary 3.7. Let $n = p^r$, where p is a prime and $r > 0$. If k is a nontrivial n-periodic knot with genus g, $\Delta_k(t) \equiv 1 \pmod p$, then $n \leq g+1$.

The corollary 3.7 is a sharpened version of Corollary 3.4 for prime power periods.

Now let $c(k)$ be the minimum crossing number of a knot k, i.e., the least number among the number of crossings in all diagrams representing the knot k. Then the following theorems follow from Corollary 3.2, Theorem 3.6, and the fact that if k is not the (c(k), 2) torus knot, then $g(k) \leq [\frac{c(k)}{2}] - 1$, where $[x]$ denote the greatest integer $\leq x$[34]:

Theorem 3.8. Let k be a knot of period n. Then $n \leq c(k) - 1$. Moreover, if $c(k)$ is odd and k is not the (c(k), 2) torus knot, then $n \leq c(k) - 2$.

Theorem 3.9. Let k be a knot of period n, $n = p^r$, p a prime, k is not the (c(k), 2) torus knot, and $[\frac{c(k)}{2}] \leq n$, Then $g_* = 0$ and exactly one of the followings holds:

(i) $n = g+1 = [\frac{c(k)}{2}]$, and $\Delta_k(t) \equiv 1 \pmod p$.
(ii) $n = g+1 = [\frac{c(k)}{2}], \deg\Delta_k(t) = 2(n-1)$, and $\Delta_k(t) \equiv (1+t+t^2)^{n-1} \pmod p$.
(iii) $n = 2g+1, \deg\Delta_k(t) = n-1$, and $\Delta_k(t) \equiv (1+t)^{n-1} \pmod p$.

The periods of prime knots with crossings ≤ 10 are completely determined by various criteria for periodicity of knots and links[4, 23]. By using his results illustrated in this section, Naik[34] gave a short proof of previously known results for periodicity of prime knots with crossings ≤ 10 and also determined the possible periods of 11-crossing knots in the table in [36]:

Theorem 3.10. (1) The only possible periods for an 11-crossing knot are $2, 3, 4, 5$, and 11.
(2) There is exactly one 11-crossing knot of period 11, namely 11_1.

(3) At most three 11-crossing knots can have period 5; there are $11_{224}, 11_{471}$, and 11_{473}.

4. The signatures of periodic links

Let F be a Seifert surface of ℓ with the genus $g(F)$. Then the homology group $H_1(F;\mathbb{Z})$ is a free abelian group with $n = 2g(F)+\mu-1$ generators. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ denote oriented simple closed curves that represent a basis for $H_1(F;\mathbb{Z})$. Consider a collar $F \times [0,1]$. For all $i, j \in \{1, 2, \cdots, n\}$, denote $\alpha_i^0 = \alpha_i \times \{0\}$ and $\alpha_i^1 = \alpha_i \times \{1\}$ with orientations induced by α_i. Define $M_F(\ell) = \emptyset$ if $g(F) = 0$ and $\mu = 1$ and, otherwise, $M_F(\ell) = (Lk(\alpha_i^0, \alpha_j^1))_{1\leq i,j\leq n}$. The matrix $M_F(\ell)$ is called the *Seifert matrix* of ℓ associated to F. Two Seifert matrices obtained from two equivalent knots or links are S-equivalent[22]. Let $\mathcal{L}$ be the set of all knots and links in S^3. The map $\sigma : \mathcal{L} \to \mathbb{Z}$ defined by $\sigma(\ell) = \sigma(M_F(\ell) + M_F(\ell)^T)$ is an invariant of knots and links and we call $\sigma(\ell)$ the *signature* of ℓ. The map $\mathcal{N} : \mathcal{L} \to \mathbb{N}$ defined by $\mathcal{N}(\ell) = \mathcal{N}(M_F(\ell)+M_F(\ell)^T)+1$ is an invariant of knots and links and we call $\mathcal{N}(\ell)$ the *nullity* of ℓ[12, 22, 30, 44, 46, 49]. There were several studies on the signatures of 2-periodic knots and links[13, 31, 25] and a certain relation between the Alexander polynomial of a knot with prime power period and its signature invariant[14]. Recently these results are extended to the more general case[24]:

Theorem 4.1. Let $\ell_1 = \ell \cup f$ be an oriented link in S^3 of $\mu + 1$ components such that f is unknotted. For any integer $n \geq 2$, let $\ell^{(n)}$ be the n-periodic covering link over ℓ_1. We assume that $\mathcal{N}(\ell^{(n)}) = \mathcal{N}(\ell)$.

(1) If either $Lk(\ell, f)$ and $\mathcal{N}(\ell)$ are odd or $Lk(\ell, f)$ and $\mathcal{N}(\ell)$ are even, then

$$\sigma(\ell^{(n)}) \equiv \begin{cases} n\sigma(\ell) & \pmod 4 \text{ if } n \text{ is odd} \\ (n-1)\sigma(\ell) + \sigma(\ell \cup f) + Lk(\ell, f) & \pmod 4 \text{ if } n \text{ is even} \end{cases}$$

(2) If either $Lk(\ell, f)$ is odd and $\mathcal{N}(\ell)$ is even or $Lk(\ell, f)$ is even and $\mathcal{N}(\ell)$ is odd, then

$$\sigma(\ell^{(n)}) \equiv \begin{cases} n\sigma(\ell) + n - 1 & \pmod 4 \text{ if } n \text{ is odd} \\ (n-1)\sigma(\ell) + \sigma(\ell \cup f) + Lk(\ell, f) + n - 2 & \pmod 4 \text{ if } n \text{ is even} \end{cases}$$

Theorem 4.2. Let ℓ be an oriented link in S^3 of μ components, let $\ell_1 = \ell \cup f$, where f is unknotted, and let $\lambda = Lk(\ell, f)$. Let $\ell^{(n)}$ be the oriented n-periodic covering link in S^3 over ℓ_1 of period $n = p^r (r \geq 1)$, where p is an odd prime. Suppose that the reduced Alexander polynomial $\tilde{\Delta}_{\ell^{(n)}}(t)$ of $\ell^{(n)}$ satisfies that

(i) $\tilde{\Delta}_{\ell^{(n)}}(t)$ is not a product of non-trivial link polynomials,
(ii) $\tilde{\Delta}_{\ell^{(n)}}(t) \not\equiv 0, \pm 1 \pmod p$.

Then

(1) $\tilde{\Delta}_{\ell^{(n)}}(t) \equiv (1 + t + \cdots + t^{\lambda-1})^{n-1} \pmod p$.
(2) If $\tilde{\Delta}_{\ell^{(n)}}(-1) \neq 0$, then

$$\sigma(\ell^{(n)}) \equiv \begin{cases} 0 & \pmod 4 \text{ if } \lambda \text{ is odd,} \\ n-1 & \pmod 4 \text{ if } \lambda \text{ is even.} \end{cases}$$

5. Finite cyclic branched covers of periodic knots

Let k be an n-periodic knot in S^3 with periodic homeomorphism $\phi : S^3 \to S^3$. From the Equivariant Tubular Neighborhood Theorem(see Chapter IV, Theorem 2.2 of [3]), there exists a tubular neighborhood T of k in S^3 such that $\phi(T) = T$. Let $X = S^3 - \text{int}(T)$ and let $\theta : X_m \to X$ be the m-fold cyclic cover of X. Then the boundary ∂X_m is a torus that covers $\partial X = T$ by wrapping the meridian of ∂X_m around the meridian of T, m-times. The m-fold cyclic cover $\mathcal{M}_m$ of S^3 branched along the periodic knot k is then obtained by attaching a solid torus $D^2 \times S^1$ to X_m along ∂X_m in such a way that $\partial D^2 \times \{\text{pt}\}$ is identified with the meridian of ∂X_m. The covering projection $\theta : X_m \to X$ clearly extends to the branched covering projection $\tilde{\theta} : \mathcal{M}_m \to S^3$. It is shown[34, Proposition 2.2] that the n-periodic homeomorphism $\phi : S^3 \to S^3$ is extended to an n-periodic homeomorphism $\tilde{\phi} : \mathcal{M}_m \to \mathcal{M}_m$ such that $\tilde{\theta} \circ \tilde{\phi} = \phi \circ \tilde{\theta}$. This implies that the cyclic group $\mathbb{Z}_n$ generated by $\tilde{\phi}$ acts on the branched cover $\mathcal{M}_m$. It is easy to see that $\mathcal{M}_m/\mathbb{Z}_n$ is the m-fold cyclic cover of S^3 branched along the factor knot $k_* = k/\mathbb{Z}_n$.

Let p be a prime integer and let $H_1(\mathcal{M}_m)_p$ and $H_1(\mathcal{M}_m/\mathbb{Z}_n)_p$ denote the p-Sylow subgroups of $H_1(\mathcal{M}_m)$ and $H_1(\mathcal{M}_m)$ consisting of the elements of order a power of p, respectively. For distinct two primes p and q, let $f_q(p)$ denote the multiplicative order of $p \pmod q$, i.e., the least positive integer such that $p^{f_q(p)} \equiv 1 \pmod q$. The following results on a characterization of p-Sylow subgroup of the torsion submodules of the homology modules of the m-fold cyclic cover $\mathcal{M}_m$ of S^3 branched along a periodic knot k were given by Naik in [34]:

Theorem 5.1. Let k be a periodic knot with a prime period q and let $H_1(\mathcal{M}_m/\mathbb{Z}_n)_p = 0$ for some prime $p \neq q$. Then there exist nonnegative integers $t, a_1, \cdots, a_t$ such that

$$H_1(\mathcal{M}_m)_p \cong (C_p)^{a_1 f_q(p)} \oplus (C_{p^2})^{a_2 f_q(p)} \oplus \cdots \oplus (C_{p^t})^{a_t f_q(p)}.$$

Corollary 5.2. Let k be a periodic knot with a prime period q and let k_* be the its factor knot. Let $\mathcal{M}_m$ and $\mathcal{M}_m^*$ be the m-fold cyclic covers of S^3 branched along k and k_*, respectively. Then for each prime factor $p \neq q$ of $|H_1(\mathcal{M}_m)|$ which does not divide $|H_1(\mathcal{M}_m^*)|$, we have that $p^{f_q(p)}$ divides $|H_1(\mathcal{M}_m)|$, where $|H_1(\mathcal{M}_m)|$ denotes the order of the torsion subgroup of $H_1(\mathcal{M}_m)$.

It is well known [4, 21] that for a primitive m-th root of unity ζ,

$$|H_1(\mathcal{M}_m)| = |\prod_{i=1}^{m} \Delta_k(\zeta^i)| \text{ and } |H_1(\mathcal{M}_m^*)| = |\prod_{i=1}^{m} \Delta_{k_*}(\zeta^i)|.$$

Theorem 5.3. Let k be a periodic knot with a prime period q and let k_* be the its factor knot. Let $\Delta_k(t)$ and $\Delta_{k_*}(t)$ be the Alexander polynomials of k and k_*, respectively. Let p be a prime such that $p \neq q$ and for a positive integer m, ζ denote a primitive m-th root of unity. Then the following statements are true.

(i) If $a \in \mathbb{Z}$ and $p \mid (\Delta_k/\Delta_{k_*})(a)$, then $p^{f_q(p)} \mid (\Delta_k/\Delta_{k_*})(a)$.
(ii) If $m \geq 2$ and $p \mid \prod_{i=1}^{m} (\Delta_k/\Delta_{k_*})(\zeta^i)$, then $p^{f_q(p)} \mid \prod_{i=1}^{m} (\Delta_k/\Delta_{k_*})(\zeta^i)$.

In 1997, under the hypothesis of Theorem 5.1, it was shown[35] that there exist nonnegative integers $t, a_1, \cdots, a_t$ such that

$$H_1(\mathcal{M}_m)_p \cong (C_p)^{2a_1 f_q(p)} \oplus (C_{p^2})^{2a_2 f_q(p)} \oplus \cdots \oplus (C_{p^t})^{2a_t f_q(p)},$$

where $f_q(p)$ denotes the least positive integer such that $p^{f_q(p)} \equiv \pm 1 \pmod{q}$. This new characterization of $H_1(\mathcal{M}_m)_p$ gives the following corollary which is more useful and easily applicable to practical examples when it is combined with the Murasugi results, Theorem 2.1 and Corollary 2.2:

Corollary 5.4 ([35]). Let k be a periodic knot with a prime period q and let k_* be the its factor knot. Let m be a power of a prime. Let $\Delta_k(t)$ and $\Delta_{k_*}(t)$ be the Alexander polynomials of k and k_*, respectively, and let ζ be a primitive m-th root of unity. Suppose that there exists a prime p such that $p \nmid \prod_{i=1}^{m} \Delta_{k_*}(\zeta^i)$, and the highest power of p which divides $\prod_{i=1}^{m} \Delta_k(\zeta^i)$ is odd. Then q is either 2 or p.

Finally we note that there exists a close relation between the Alexander polynomial of a knot and the genus of a knot. As an example, H. Seifert[42] showed that for any knot polynomial Δ, there exists a knot k with $g(k) = \frac{1}{2}\mathrm{deg}\Delta$ and with $\Delta = \Delta_k(t)$. Using Theorem 3.5 and Theorem 5.3, Naik[34] proved the following theorem which discuss the genus of a periodic knot with a prescribed Alexander polynomial and show that for a certain knot polynomial Δ with $\Delta \equiv 1 \pmod{q}$ for some $q > 1$, the genus of any q-periodic knot k with $\Delta_k = \Delta$ has to be fairly high:

Theorem 5.5. Let Δ be a nontrivial knot polunomial and let q be a prime. Suppose that for each knot polynomial $f(t) \neq \Delta$ and dividing Δ, there exists an integer $a = a(f)$ and a prime $p = p(f)$ satisfying the following three conditions:

$$\text{(i) } p \neq q, \ \text{(i) } p \mid (\Delta/f)(a), \ \text{(i) } p^{f_q(p)} \nmid (\Delta/f)(a).$$

Then for any period q knot k with $\Delta_k = \Delta$, we have that $g(k) \geq \frac{1}{2}q\mathrm{deg}\Delta$.

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