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ON DECOMPOSITION THEOREMS

NOBUYUKI ODA

Introduction

In a series of works, Eckmann and Hilton discovered "**duality in homotopy theory**" in the mid-fifties of the 20th century. In the principle of **Eckmann-Hilton duality**, the product space $A \times B$ and the one point union (or wedge sum) $A \vee B$ of spaces A and B with base point are dual concepts, for example. Under some conditions, a case can happen that a space X is decomposed into a product or a one point union of other spaces Y and Z as

$$X \simeq Y \times Z \quad \text{or} \quad X \simeq Y \vee Z,$$

where $\simeq$ means the base point preserving homotopy equivalence. When one considers fibre sequence $F \to E \to B$, a question arises: Is $E \simeq F \times B$? (in many cases, this question is answered by showing that there exists an induced isomorphism of homotopy groups). Dually, when one considers co-fibre sequence $A \to X \to C$, there arises a dual question: Is $X \simeq A \vee C$? (dually, this question is answered by showing that there exists an induced isomorphism of cohomology groups). The present article is a very short survey on the theorems concerning decompositions of spaces of the types mentioned above. It is not a complete survey and mainly those results are considered which are closely connected with a joint work with Norio Iwase: "Decompositions of fibrations and cofibrations" [32].

This article contains the following sections:

1 Eckmann-Hilton duality and decomposition of spaces
2 Rational Hopf spaces and co-Hopf spaces
3 Ganea conjecture
4 Conner-Raymond splitting theorem
5 Splitting off Hopf spaces and co-Hopf spaces
6 Decompositions of fibrations and cofibrations

In the first section, we see some of the results obtained in the early stages of the research of Eckmann-Hilton duality. In the second section, decompositions of rational spaces are considered. Some complete results are obtained for rational spaces partly beacause the homotopy groups and homology groups of them are rational vector spaces. Next two sections deal with questions of splitting off a

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wedge sum of circles and a product of circles respectively. More precisely, in the third section we consider the following Ganea's question: For any (non-simply connected) co-Hopf space X, does there exist a splitting $X \simeq Y \vee S$, where Y is a 1-connected space and S is a wedge of circles S^1 ? ; and in the fourth section, we consider the following question: For a space X, does there exist a splitting $X \simeq Y \times T$, where T is a product of S^1 ? In the fifth section, we consider splitting theorems of Dula and Gottlieb [12]. These theorems are very general results on splitting off Hopf spaces and co-Hopf spaces. In the final section, we mention a general theorems of decompositions of fibrations and cofibrations proved in [32].

In the following sections, the symbol . . . in the quotations means that some words or sentences are omitted.

1. Eckmann-Hilton duality and decomposition of spaces

The concept of Hopf space goes back to [28] (1939) at least (cf. [29], 1941), and the terminology "*H-space*" was used by Serre [44] (1951). However, we can see the notions of **Hopf space** (H**-space**) and **co-Hopf space** (H'**-space**) as dual concept in [13] (1957) and [14] (1959). It is also proved that *if A is a co-Hopf space and B is a Hopf space, then the two group structures of the homotopy set $[A, B]$ coincide and abelian* (Eckmann and Hilton [14], 1959).

Ganea and Hilton [19] (1959) considered decompositions of spaces into Cartesian products and one point unions in the light of duality: The present paper is concerned with particular cases, obtained by suitably restricting the spaces involved, of the following general problem.

Given a topological space X, we ask whether there exist integers $n \geq 2$ and non-contractible spaces $X_1, \ldots, X_n$ such that X has the homotopy type of the Cartesian product $X_1 \times \cdots \times X_n$ or of the union $X_1 \vee \cdots \vee X_n$.

(Ganea and Hilton [19], 1959)

In the paper quoted above, Ganea and Hilton used the **Lusternik-Schnirelmann category** and **cocategory**. Browder [9] (1959) determined the structure of cohomology ring of covering spaces of H-spaces. Eckmann and Hilton [15] (1960) used **operators** and **cooperators** for decomposition of spaces:

. . . In the final section we consider the theorem of Spanier-Whitehead [5] which asserts that *if the fibre F of the fibration $p : X \to Y$ is contractible in X, then F is an H-space* (space with multiplication). We show how a contraction of F determines a definite homotopy equivalence of $\Omega X \times F$ with ΩY and construct a homotopy inverse. The dual theorem then asserts that *if the projection $q : X \to F$ from the total space X of the cofibration $i : Y \to X$ onto the cofibre F is contractible then there is a homotopy equivalence of ΣY with $\Sigma X \vee F$. As a consequence F is an H'-space* (space with comultiplication).

(Eckmann and Hilton [15], 1960)

Hilton's book Homotopy theory and duality ([25], 1965) contains many duality arguments, and there is a table of dual constructions in homotopy theory in Remark 3 on p.122. There are following arguments on p.103 about **decompositions of co-Hopf spaces and Hopf spaces** using **cofibration** and **fibration** respectively: Let $j : X \to X'$ be a cofibration with the cofibre F and $q : X' \to F$ the projection. Suppose that there is a map $p : X' \to X$ with $p \circ j = 1_X$. If X' admits a comultiplication, then $q + p : X' \to F \vee X$ can be defined and induces homology isomorphisms $H_*(X') \approx H_*(F \vee X)$. In the dual situation, let $p : Y \to X$ be a fibre map with fibre F and inclusion $i : F \to Y$. We suppose that there is a section $j : X \to Y,\ \ p \circ j = 1_X$. If Y admits a multiplication, then we may define a map $j \circ p_1 + i \circ p_2 : X \times F \to Y$ which induces an isomorphism $\pi_*(X \times F) \to \pi_*(Y)$.

Some other general references for Hopf spaces and co-Hopf spaces are Topics in the homology theory of fibre bundles ([8], 1967) by Borel, Hopf spaces ([48], 1976) by Zabrodsky, The homology of Hopf spaces ([34], 1988) by Kane and Co-H-spaces ([3], 1995) by Arkowitz.

2. Rational Hopf spaces and co-Hopf spaces

Thom [45] (1956), Berstein [6] (1961) and Curjel [11] (1963) studied **mod $\mathcal{C}$ Hopf space and co-Hopf space**. Arkowitz and Curjel [4] characterized H-Raum mod $\mathcal{F}$ (1965).

Toomer [46] (1975) proved the following decomposition theorem: Theorem 7. *If X is a rational space, $\Sigma\Omega X$ has the homotopy type of a wedge of rational spheres.*

(Toomer [46], 1975)

Baues [5] (1977) used "**instabilen CW-Komplex**" for the study of rational homotopy type:

SATZ : *Sei X ein einfach zusammenhängender CW-Raum. Dann gibt es einen instabilen CW-Komplex $\overline{X}$ und eine Abbildung $h : \overline{X} \to X$, welche eine rationale Homotopieäquivalenz ist.*

(2.1) SATZ : *Sei Σ ein einfach zusammenhängender Co-H-Raum, so ist der Hurewicz-Homomorphismus $h : \pi_*(\Sigma) \otimes Q \to H_*(\Sigma, Q)$ surjektiv.* (Vergl. [5] und [7]).)

(2.2) KOROLLAR : *Sei Σ ein einfach zusammenhängender CW- und Co-H-Raum. Sei $B \subset H_*(\Sigma, Q)$ eine Basis. Dann gibt es eine Abbildung $\beta : \bigvee_{b \in B} S^{|b|} \to \Sigma$, welche eine rationale Homotopieäquivalenz ist.*

(Baues [5], 1977)

Henn [24] (1983) proved a decomposition of **almost rational co-H-spaces**. He used the fact that ΩX is a product of Eilenberg-Mac Lane spaces for any 1-connected rational space X and the Ganea's result [17] (1970) that any co-Hopf space X is a retract of $\Sigma\Omega X$: Now we remark that for 1-connected rational spaces X a decomposability as a wedge of rational spheres is equivalent to the Hurewicz-homomorphism h_X being an epimorphism. Using this and passing to the limit shows that $SK(\pi_1(\Omega X),1)$ decomposes in the case of an arbitrary $\pi_1(\Omega X)$, too. . . .

3. Theorem. *If X is a 0-connected almost rational co-H-space then X is up to homotopy a wedge of rational spheres of dimension at least two and of ordinary one-dimensional spheres.*

(Henn [24], 1983)

Scheerer [42] (1985) proved decomposition theorems of rational Hopf spaces and co-Hopf spaces **without assuming the finite type assumptions**:

The following investigation has been motivated by the remarks of [2] (chap. VI, (4.6)) on rationalized loop spaces and suspensions. We think it worthwhile to derive these results in a different, easier way for $\mathbf{Q}$-local H-spaces and co-H-spaces as well as to get rid of the finite type assumptions. In an appendix we will indicate that the view displayed here on the rational situation might also be obtained on the more general situation of decomposable H- and co-H-spaces.

. . .

Lemma: *A $\mathbf{Q}$-local H-space is homotopy equivalent to a weak product of Eilenberg-Mac Lane spaces.*

Note: This fact is well known (see [21], [14], Proposition 2 or [7], Satz 10.6.). We will indicate a short proof which seems to be "folklore" and which, in different terms, has already been given in [7], Satz 10.6. It also relates to [2], Chap. V, (3.10). . . .

Lemma: *Let C be a co-H-space in* 1-$\mathrm{CW}_{\mathbf{Q}}$. *Then C is homotopy equivalent to a wedge of $\mathbf{Q}$-local spheres.*

(Scheerer [42], 1985)

Scheerer [43] proved decomposition theorems for subrings of Q (1986). McGibbon and Wilkerson [36] studied loop spaces of finite complexes at large primes (1986).

We see that decompositions of rational Hopf spaces and co-Hopf spaces were studied until this time. What happens for rational spaces which are not Hopf spaces nor co-Hopf spaces? Oprea [38] (1986) obtained the following type of decomposition theorems for general rational spaces. We see that the **Gottlieb group** and the **Hurewicz map** play important roles in the proof. The Gottlieb group was introduced by Gottlieb [20] (1965) and [21] (1969). The reference [FT] in Theorem B is Felix and Thomas [16] (1981):

Theorem A [H, H1, FT] *If $F \to E \xrightarrow{p} B$ is a fibration and $\partial_\#$ is surjective, then F has the homotopy type of a product of Eilenberg-Mac Lane spaces.*

. . . The role of Eckmann-Hilton duality in topology provides a secondary theme for this paper. There is an interaction between the connecting homomorphism in the homology sequence of a cofibration and the Hurewicz map of the cofibre which dualizes the situation alluded to earlier for the case of a fibration. Before we discuss the effects of these dual relationships, we note that Theorem A (and its proof) may be dualized to yield,

THEOREM B [FT] *If* $X \xrightarrow{f} Y \to C$ *is a cofibration and* ∂_* *is injective, then* C *has the homotopy type of a wedge of spheres.*

. . . We now come to the decomposition results which form the basis of this investigation. Note the heavy dependence on the $\partial_\#$-h and ∂_*-h interaction. In order to understand the statements of the theorems, we remind the reader that, for any X subject to our conventions, ΩX has the type of a product of Eilenberg-Mac Lane spaces and ΣX has the type of a wedge of spheres. Also, we denote the image of the Hurewicz map $h : \pi_*(X) \to H_*(X)$ by $S(X)$.

THEOREM 1. *If* $F \to E \to B$ *is a fibre sequence, then*

$$F \simeq \mathcal{F} \times K$$

where K *is the subproduct of* ΩB *maximal with respect to the conditions,* $\pi_*(K) \cap \operatorname{Ker} \partial_\# = 0$ *and* $\partial_\#(\pi_*(K)) \cap \operatorname{Ker} h = 0$.

THEOREM 2. *If* $X \to Y \to C$ *is a cofibre sequence, then*

$$C \simeq \mathcal{C} \vee S$$

where S *is the maximal subwedge of* ΣX *such that* $\partial_*|_{S(C)}$ *is surjective.*

(Oprea [38], 1986)

Applications were studied in Oprea [39] (1987). Oprea [40] (1988) studied p-local cases.

In [2] (1987) Aguadé call a 0-connected spaces X a T**-space** if the fibration

$$\Omega X \longrightarrow \Lambda X \longrightarrow X$$

is trivial, in the sense of fibre homotopy type, where ΛX is the space of free loops on X and ΩX is the space of based loops on X. Hopf space is a T-space. He also defined the notion of T-maps and T_n-spaces and generalized a result of Aguadé [1] (1981). He also obtained results on **rational** T**-spaces**: Assume we have a Postnikov system for X.

PROPOSITION 3.1. *If* X *is a* T*-space and* $n \geq 1$*, then there exists a* T*-structure on* X_n *such that* $j_n : X \to X_n$ *is a* T*-map.*

THEOREM 3.2. *If* X *is a* T*-space, then the spaces of any Postnikov system for* X *are* T*-spaces and the Postnikov invariants are* T*-maps.*

THEOREM 3.3. X *is a rational* T*-space if and only if* X *has the same rational homotopy type as a product of Eilenberg-Mac Lane spaces.*

(Aguadé [2], 1987)

3. Ganea conjecture

Ganea [18] posed the following question: **Problem 10**: *Is any (non-simply connected) co-Hopf space of the homotopy type of* $S^1 \vee \cdots \vee S^1 \vee Y$*, where there may be infinitely many circles and* $\pi_1(Y) = 0$ *?*

(Ganea [18], 1971)

The problem quoted above is known as a Ganea conjecture. To answer Ganea's problem, Berstein and Dror [7] (1976) studied **associative co-operation** and Hilton, Mislin and Roitberg [26] (1978) studied **coloop strcture**. They proved the following results:

. . . This characterization is given in terms of "co-operation" of a wedge of circles S on X.

Definition 1.1. We say that a wedge of circles S co-operates on X if there is a map $\psi : X \to X \vee S$ such that $p \circ \psi \sim 1_X$ and that $q \circ \psi$ induces an isomorphism on fundamental groups.

Recall that p and q are the natural projections on the first and second factors of a wedge. In particular $G = \pi_1 X$ must be free.

Definition 1.2. We say that a co-operation is associative if

$$(\psi \vee 1) \circ \psi \sim (1 \vee \phi) \circ \psi$$

where ϕ is some associative co-multiplication on S.

Theorem 1.5. *A necessary and sufficient condition for a space* X *to admit a representation* (0.1) *as a wedge between a simply connected space and a wedge of circles is the existence of an associative co-operation of* S *on* X*.*

(Berstein and Dror [7], 1976)

We show in Section 4 that the Ganea conjecture holds if Y is a *coloop*; we give the explicit definition of a coloop in Section 2, but, in fact, coloops Y are characterized by the property that $[Y, A]$ is a loop for all spaces A. . . .

. . . Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{\Sigma f} \cdots \tag{4.2}$$

be the Puppe sequence of f. We prove

Proposition 4.1. *Let* $f : X \to Y$ *be a co-H-map. Then, in* (4.2)*, we may give* Z *the sturcture of a co-H-space in such a way that* g *is a co-H-map. If, further,* f *is coretractile,* Y *is a coloop, and* Z *is* 1*-connected, then the co-H-structure on* Z *is determined by the requirement that* g *be a co-H-map.*

Theorem 4.2. *Let* $f : X \to Y$ *be a co-H-map of coloops with the mapping cone* Z 1*-connected. If* f *has a left inverse then* $Y \simeq Z \vee X$*.*

Corollary 4.3. *Let Y be a connected coloop. Then $Y \simeq Z \vee B$, where Z is a 1-connected co-H-space and B is a bunch of circles.*

(Hilton, Mislin and Roitberg [26], 1978)

Recent progress in the study of Ganea conjecture (Problem 10) is due to Iwase [31] (2001), Iwase, Saito and Sumi [33] (1999), Hubbuck and Iwase [30] (2002).

4. Conner-Raymond splitting theorem

We can see the following result on **splitting off a product of circles** from H-spaces in a paper by Hilton and Roitberg [27] (1970) :

We first dispose of the cases $(q, n) = (1,1),\ (1,3),\ (1,7)$ by means of the following theorem whose proof we omit; the observation was made to us by G. Mislin.

2.1 Theorem. *Let X be an H-space with $\pi_1(X)$ free abelian of rank k. Then we have a fibration $\widetilde{X} \to X \to (S^1)^k$ in which the maps are H-maps, and this fibration has a cross-section. Thus $X \simeq \widetilde{X} \times (S^1)^k$.*

(Hilton and Roitberg [27], 1970)

Conner and Raymond studied T^k**-action** on manifolds. **Conner-Raymond splitting theorem** was proved in a paper published in 1971 :

. . . A factor of a central extension $0 \to Z^k \xrightarrow{\alpha} \pi \xrightarrow{\beta} N \to 1$ is a pair (L, ϕ) consisting of an epimorphism $\phi : \pi \to L \to 1$ for which the composition

$$\alpha_{\#} = \phi \circ \alpha : Z^k \to L$$

is a monomorphism. . . .

An action of a toral group (T^k, X) is *injective* if and only if at each $x \in X$ the map $f^x : T^k \to X$ given by $f^x(t) = tx$ induces a monomorphism ${f_*}^x : \pi_1(T^k, 1) \to \pi_1(X, x)$. It is readily seen that in an injective action the isotropy subgroups are all finite.

A *factor* (L, ϕ) *of the injective action* (T^k, X) is simply a factor of the associated central extension

$$0 \to Z^k \xrightarrow{{f_*}^x} \pi_1(X, x) \to N \to 1.$$

An important factor to keep in mind is $(\pi_1(X, x), \text{identity map})$.

If (T^k, X) is an injective action so that (T^k, X) is equivariantly homeomorphic to $(T^k, T^k \times Y)$ with the latter action just left translation on the first factor and trivial on the Y factor we say *the action* (T^k, X) *splits*.

In [3; §4] we defined the *lifting* of actions of pathwise connected groups G to covering spaces B_H for which H contains $\mathrm{im}\,({f_*}^x)$, where

$${f_*}^x : \pi_1(G, e) \to \pi_1(X, x),$$

and B_H is the covering space of X corresponding to the subgroup H. If H is normal the lifted action (G, B_H) covering (G, X) commutes with the covering transformations $\pi_1(X, x)/H$.

If (T^k, X) is an injective action, and H is a normal subgroup, $H \supseteq \operatorname{im} f_*{}^x$, so that the covering action (T^k, B_H) splits, then we say that the action (T^k, X) *splits on* B_H, or simply that (T^k, B_H) *is a splitting action for* (T^k, X).

3.1 SPLITTING THEOREM. *Let (T^k, X) be an injective action and (L, ψ) a factor. If H is the kernel of $\pi_1(X, x) \to L/\operatorname{im} f_\sharp{}^x$, then the action (T^k, X) splits on B_H.*

. . .

4.2 THEOREM. *Let (T^k, X) be an action on a space for which $H_1(X; Z)$ is finitely generated. If $f_* : H_1(T^k; Z) \to H_1(X; Z)$ is a monomorphism then for a suitable value of n the space X can be fibered over T^k with structure group $(Z_n)^k$.*

(Conner and Raymond [10], 1971)

Lück [35] (1987) studied the **Gottlieb group $G_1(X)$ to prove a splitting theorem** (cf. Gottlieb [22] (1989) p.217, Oprea [41] (1990) p.220):

Proposition 4.3. *Let X be a CW-complex. There exists a CW-complex Y with $X \simeq Y \times S^1$ if and only if $\pi_1(X)$ can be written as $G \times \mathbb{Z}$ with $\mathbb{Z} \subset G_1(X)$.*

(Lück [35], 1987)

Gottlieb [22] (1989) defined the **toral number** and the **Hurewicz rank** to split off tori (cf. Halperin [23], 1985): In this paper all spaces are CW complexes. All spaces are assumed to be connected. We say that X has *toral number* n if n is the largest integer such that X is homotopy equivalent to $Y \times T^n$ for some space Y. Note that Y is necessarily homotopically equivalent to a covering space of X.

Let G be a subgroup of $\pi_1(X)$. Define the *Hurewicz rank of* G as follows. Consider the image of G under the Hurewicz homomorphism h in the homology group. Then $h(G)$ may contain free summands of $H_1(X)$. We say the *Hurewicz rank of* G is the maximum rank of these free summands. If there is no free summand in $h(G)$ then we say the Hurewicz rank of G is zero and if there is no maximum we say the Hurewicz rank of G is infinite.

THEOREM. *The toral number of X is equal to the Hurewicz rank of $G_1(X)$.*

COROLLARY 1. *There exists a finite covering $\widetilde{X}$ of X whose toral number is greater than or equal to the rank of $h(G_1(X))$.*

COROLLARY 2. *The dimension of X is greater than or equal to the rank of $G_1(X)$.*

COROLLARY 3. *If F is dominated by a finite complex and is the fibre of a fibration with simply connected total space, then F is homotopically equivalent to a finite complex except possibly when F has a nontrivial finite fundamental group.*

COROLLARY 4. (Conner-Raymond Splitting Theorem [C-R]). *If a torus T^k acts on X so that evaluation at a point gives a map $\omega : T^k \to X$ so that $\omega_*(H_1(T^k))$ is a direct summand of $H_1(X)$ of rank k, then X is equivariantly homeomorphic to $T^k \times Y$ for some space Y where T^k acts on the product by $g(h, y) = (gh, y)$.*

(Gottlieb [22], 1989)

We quote the following results by Oprea [41] (1990), where the **Gottlieb group** and the **Hurewicz image** are used effectively: THEOREM 6. *Let X be a space with $H_1(X;\mathbb{Z})$ finitely generated. If there exists $\alpha \in G(X)$ with Hurewicz image $h(\alpha)$ of infinite order, then there is a finite cyclic cover $\overline{X}$ of X with $\overline{X} \simeq Y \times S^1$.*

COROLLARY 8. *If X is an H-space and $H_1(X;\mathbb{Z})$ has a $\mathbb{Z}$-summand, then $X \simeq Y \times S^1$.*

THEOREM 10. *If $h(G(X))$ contains a free summand of $H_1(X;\mathbb{Z})$ of rank n, then $X \simeq Y \times T^n$.*

(Oprea [41], 1990)

5. Splitting off Hopf spaces and co-Hopf spaces

Dula and Gottlieb [12] (1990) proved a very general results to split off Hopf spaces and co-Hopf spaces. The **generalized Gottlieb set** or the **Varadarajan set** ([47], 1969) was used for the decomposition of fibrations. Dually, the **generalized dual Gottlieb set** or the **dual Varadarajan set** ([47], 1969) was used for the decomposition of cofibrations:

Conner and Raymond studied the action of the torus $T^k = S^1 \times S^1 \times \cdots \times S^1$ on a T^k-space. The *orbit map* $w : T^k \to X$ is induced by the action of T^k on a base point in X. The Conner-Raymond Splitting Theorem is first stated and proved in [CR, Theorem 3.1] and is restated and proved in [G3, Corollary 4] in the following form:

THE CONNER-RAYMOND SPLITTING THEOREM. *If a torus T^k acts on X so that evaluation at a point gives a map $w : T^k \to X$ so that $w_*(H_1(T^k))$ is a direct summand of $H_1(X)$ of rank k, then X is equivariantly homeomorphic to $T^k \times Y$ for some space Y where T^k acts on the product by $g(h,y) = (gh,y)$.*

The above statement will be referred to as the Conner-Raymond Splitting Theorem. This is a particular case of the original version [CR], in which the factor (L,ϕ) is the identity map of $\pi_1(X,x)$, the kernel H is trivial and π_1 is replaced by $H_1(\ ,Z)$.

Oprea [O, Theorem 11] has independently of Gottlieb found a Conner-Raymond Splitting Theorem for the case that k equals 1.

In this paper we prove the following generalization of the Conner-Raymond Splitting Theorem. G is assumed to be a compact Lie group. X is a completely regular pathconnected G-space and $w : G \to X$ is the orbit map. X and G are assumed to be homotopy equivalent of CW complexes. $\underline{G}$ is the identity map of G. Let T_G be the functor from the homotopy category to the category of groups, sending a space X to the group $[X,G]$, and $w^\sharp : [X,G] \to [G,G]$ be the morphism induced by precomposing with w, sending a homotopy class $f : X \to G$ to $f \circ w : G \to G$.

Theorem 3.1. *The following statements are equivalent:*
(i) *X is isomorphic as a G-space to $G \times (X/G)$, where G acts diagonally, by multiplication on the first factor and trivially on the second.*
(ii) *$w^\sharp$ has a right inverse, that is $w^\sharp$ is onto.*
(iii) *w has a left inverse $r : X \to G$.*

In the particular case $G = T^k$, Theorem 3.1 implies the Conner-Raymond Splitting Theorem, as explained in 3.2 below.

A proof of Theorem 3.1 follows from Theorem 1.3 below. This is a splitting theorem which characterizes when a given space is a cartesian product of an H-space. Again all spaces involved are assumed to be homotopy equivalent to CW complexes.

1.3. Splitting Theorem. *Given spaces X, K and Y the following statements are equivalent:*
(i) *K is an H-space and there exists a space Y such that X is homotopy equivalent to $K \times Y$.*
(ii) *There is a class $i : K \to X$ in the generalized Gottlieb set $G(K, X)$ (as defined by Varadarajan* [V]*) such that $i^\sharp$ has a right inverse (that is $i^\sharp$ is onto).*
(iii) *There are classes i in $G(K, X)$ and $r : X \to K$ such that r is a left inverse for i (and has a fiber Y).*

. . . X and K are homotopy equivalent to connected and simply connected CW complexes in the following Theorem.

A necessary and sufficient condition for splitting co-H-spaces

2.2 Theorem. *The following conditions are equivalent:*
(i) *K is a co-H-space and there exists a simply connected space Y such that X is homotopy equivalent to $K \vee Y$.*
(ii) *There is a class i in $DG(X, K)$ such that $i_\sharp : [K, X] \to [K, K]$ has a right inverse, that is $i_\sharp$ is onto.*
(iii) *There are classes i in $DG(X, K)$ and $r : K \to X$ such that r is a right inverse of i (and has a cofiber Y).*

(Dula and Gottlieb [12], 1990)

6. Decompositions of fibrations and cofibrations

Now we consider the problem in Introduction: For a **fibre sequence** $F \to E \to B$, under what conditions is there a homotopy decomposition $E \simeq F \times B$? Dually, for a **co-fibre sequence** $A \to X \to C$, under what conditions is there a homotopy decomposition $X \simeq A \vee C$? We quote the following general results obtained in [32]:

Theorem. *Let D, X and B be* 0*-connected spaces. Let $D \xrightarrow{i} X \xrightarrow{p} B$ be a fibre sequence with a homotopy cross section $s : B \to X$. If there is a pairing $\zeta : D \times B \to X$ with axes (i, s), then ζ is a weak homotopy equivalence $D \times B \simeq_w X$.*

Theorem (dual). *Let $A \xrightarrow{i} X \xrightarrow{q} C$ be a co-fibre sequence with a homotopy retraction $r : X \to A$. If there is a copairing $\xi : X \to A \vee C$ with coaxes (r, q), then ξ is a co-weak homotopy equivalence $X \simeq'_w A \vee C$.*

In Theorem above, **weak homotopy equivalence** $D \times B \simeq_w X$ means that the induced homomorphism $\zeta_* : [V, D \times B] \to [V, X]$ is an isomorphism for any co-grouplike co-Hopf space V. Dually, in Theorem (dual), **co-weak homotopy equivalence** $X \simeq'_w A \vee C$ means that the induced homomorphism $\xi^* : [A \vee C, K] \to [X, K]$ is an isomorphism for any grouplike Hopf space K. These theorems are proved by making use of "**generalized square lemma**" (Theorem 2.7 of [37]).

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Fukuoka University, Dept. of Applied Mathematics