Trends in Mathematics
Information Center for Mathematical Sciences
Volume 7, Number 1, June, 2004, Pages 49–56
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CALCULATION OF THE COHOMOLOGY OF A FUNCTION SPACE VIA THE EILENBERG-MOORE SPECTRAL SEQUENCE

KATSUHIKO KURIBAYASHI

1. Introduction

Let X and Y be topological spaces and let $\mathcal{F}(X,Y)$ be the function space of continuous maps from X to Y with the compact-open topology. We denote by $\mathcal{F}_*(X,Y)$ the subspace of consisting of base-point-preserving maps when X and Y are based spaces. The free loop space, written LX, is the function space of the form $\mathcal{F}(S^1,X)$. In [17] and [18], Smith has investigated the structure of the cohomology algebra of LX for which X is simply-connected by using the Eilenberg-Moore spectral sequence (EMSS) associated with the homotopy fibre square (1.1):

$$\begin{array}{ccc} LX & \longrightarrow & X \\ {\scriptstyle ev_*}\downarrow & & \downarrow{\scriptstyle \Delta} \\ X & \xrightarrow[\Delta]{} & X \times X, \end{array}$$

where Δ is the diagonal map and ev_* is the evaluation map at zero. Subsequently, the author and Yamaguchi [11] have shown that the EMSS is also reliable in computing the cohomology $H^*(LX;\mathbb{F}_p)$ by determining explicitly the algebra structure of the cohomology provided X is a simply-connected space whose mod p cohomology is generated by one element.* Recently, Kono and the author [7] have shown that, when $H^*(X;\mathbb{F}_p)$ is a polynomial algebra, the cohomology algebra $H^*(LX;\mathbb{F}_p)$ over the Steenrod algebra can be represented exactly by that of $H^*(X;\mathbb{F}_p)$ and the module derivation $\mathfrak{D} : H^*(X;\mathbb{F}_p) \to H^{*-1}(LX;\mathbb{F}_p)$ via the EMSS mentioned above. We refer to [8] for the definition of the module derivation and its fundamental properties. Let G be a compact connected Lie group and let $P \to M$ be a G-bundle over a manifold M. Let $\mathcal{G}(P)$ denote the gauge group, namely, the space of sections of the adjoint bundle $P \times_{ad} G \to M$. The result[1, Proposition 2.4] due to Atiyah and Bott asserts that the classifying space $B\mathcal{G}(P)$ is homotopy equivalent to $\mathcal{F}_f(M,BG)$ the component of the function space $\mathcal{F}(M,BG)$ including the classifying map f of the given bundle. Suppose that M is a connected manifold with dimension ≤ 3 and G is simply-connected. Since BG is 4-connected, it follows that $P \to M$ is the trivial bundle and hence $B\mathcal{G}(M \times G) \simeq \mathcal{F}(M,BG)$. The fact motivates us to

*This is a survey of the article [10]. Proofs are generally omitted here.

*Nodombol and Thomas have also made the calculation using the Hochschild homology of $H^*(X;\mathbb{F}_p)$ in [15]. Dupont and Hess [4] and Menichi [14] have considered the cohomology algebra of such a free loop space from the viewpoint of algebraic models for spaces.

investigate the function whose source space M is of 2 or 3-dimensional. When M is a surface, there are a few results concerning the cohomology of the function space $\mathcal{F}(M, BG)$. Masbaum [12] [13] has determined explicitly the integral cohomology algebras of $\mathcal{F}(S^2, BSU(2))$ and $\mathcal{F}(\Sigma_g, BSU(2))$. Here Σ_g denotes the Riemann surface of genus g. By applying fiberwise homotopy theory, Crabb [3] has computed the integral cohomology algebra $\mathcal{F}(S^2, BSU(m))$.

Our interest here lies in the study of the function space $\mathcal{F}(M, BG)$ in which M is a closed orientable 3-dimensional manifold and BG is the classifying space of a simply-connected Lie group G. More precisely, in this manuscript, we consider the cohomology of the function space $\mathcal{F}(M, BG)$ with coefficients in a Noetherian ring R. I presume that it is difficult to determine exactly the cohomology of $\mathcal{F}(M, BG)$ even if the coefficient is a field. We then confine ourself to considering the cohomology $H^*(\mathcal{F}(M, BG); R)$ for low degrees. In order to see what information we can get from the cohomology in low degrees, we first consider the function space from the viewpoint of rational homotopy theory; that is, rational homotopy types of the function spaces are studied.

Applying the construction of minimal models for function spaces due to Brown and Szczarba [2] or that due to Haefliger [6] to our context, we have a minimal model for the function space $\mathcal{F}(M, BG)$: $(\wedge Z, 0)$, $Z^l = \oplus_{j-i=l} V^j \otimes H_i(M; \mathbb{Q})$, where $(\wedge V, 0)$ is a minimal model for BG. It is not hard to deduce the following facts from the minimal model for $\mathcal{F}(M, BG)$.

FACT 1. *Let M and M' be surfaces or 3-dimensional manifolds. Then $\mathcal{F}(M, BG)$ is homotopy equivalent to $\mathcal{F}(M', BG)$ after localization at zero if and only if $H_*(M; \mathbb{Q}) \cong H_*(M'; \mathbb{Q})$ as a vector space.*

FACT 2. *The first rational Betti numbers b_1 of orientable closed 3-dimensional manifolds M completely classify the rational homotopy types of the function spaces of the form $\mathcal{F}(M, BG)$.*

Moreover we are aware that the $\mathbb{Q}$-cohomology of $\mathcal{F}(M, BG)$ in degree 3 has the first *homology* $H_1(M; \mathbb{Q})$ as a direct summand.

One may expect similar results hold in the case where the coefficient is a field with positive characteristic, more general a ring R.

Expectation. (i) For an orientable manifold M, the homology $H_1(M; R)$ appears in the cohomology $H^3(\mathcal{F}(M, BG); R)$ as a direct summand.
(ii) The first mod p Betti numbers of orientable 3-dimensional manifolds is also topological invariants for function spaces of the form $\mathcal{F}(M, BG)$ when fixing the target space BG.

Our main theorem below asserts that our expectation comes true.

Theorem 1.1. [10, Theorem 1.2] *Suppose that $\mathrm{Tor}_{\mathbb{Z}}(\mathbb{Z}/2, R) = 0$. Let M be a closed orientable 3-dimensional manifold and BG the classifying space of a compact simply-connected Lie group G whose integral cohomology is torsion free. Put $H^i = H^i(\mathcal{F}(M, BG); R)$. Then there exists a short exact sequence*

$$0 \to H_1(M; R)^{\oplus r} \oplus R^{\oplus s} \oplus H^1 \otimes H^2 \xrightarrow{\alpha} H^3 \to (R/2R)^{\oplus r} \to 0$$

such that $\alpha|_{H^1\otimes H^2}$ *is the cup product, where* $r = \mathrm{rank} H^4(BG)$, $s = \mathrm{rank} H^6(BG)$. *Moreover* H^1 *is a free* R*-module for any* R, *and* H^2 *is also free if* R *is a PID.*

Corollary 1.2. [10, Corollary 1.3] *Let* p *be an odd prime number. If* $H^i(\mathcal{F}(M, BG); \mathbb{F}_p)$ *is isomorphic to* $H^i(\mathcal{F}(M', BG); \mathbb{F}_p)$ *as a vector space for* $i \le 3$, *then the first Betti number of* M *with the coefficient* $\mathbb{F}_p$ *coincides with that of* M'.

In the next section, we introduce an EMSS converging to the cohomology of a function space whose source space is an adjunction space. It seems that the EMSS is relevant to the study of such a function space. In Section 3, we give an outline of the proof of Theorem 1.1. It is important to mention here that ingredients for proving the theorem are the EMSS associated to a Heegaard splitting of M and the fundamental group $\pi_1(M)$ represented by the Heegaard diagram.

2. The EMSS converging to a function space whose source is an adjunction space

The adjunction space $Y \cup_\alpha X$, which is obtained by attaching X to Y along α, defines the push out diagram

$$\begin{array}{ccc} Y \cup_\alpha X & \xleftarrow{\alpha_X} & X \\ {\scriptstyle i_Y}\uparrow & & \uparrow{\scriptstyle i} \\ Y & \xleftarrow[\alpha]{} & A. \end{array}$$

For a given space Z, the functor $\mathcal{F}(\ , Z)$ maps a topological space X to the function space $\mathcal{F}(X, Z)$. A map $f : A \to Y$ induces the morphism $f^\natural = \mathcal{F}(f, id_Z) : \mathcal{F}(Y, Z) \to \mathcal{F}(A, Z)$ defined by $\mathcal{F}(f, id_Z)(u)(a) = uf(a)$ for $a \in A$ and $u : Y \to Z$. Applying the functor $\mathcal{F}(\ , Z)$, we have a commutative square

$$\begin{array}{ccc} \mathcal{F}(Y \cup_\alpha X, Z) & \xrightarrow{\alpha_X^\natural} & \mathcal{F}(X, Z) \\ {\scriptstyle i_Y^\natural}\downarrow & & \downarrow{\scriptstyle i^\natural} \\ \mathcal{F}(Y, Z) & \xrightarrow[\alpha^\natural]{} & \mathcal{F}(A, Z). \end{array}$$

If $i^\natural$ is surjective, then so is $i_Y^\natural$. Therefore $i^\natural$ and $i_Y^\natural$ are fibrations and hence the commutative diagram is regarded as a pullback diagram of the map $i^\natural$ along $\alpha^\natural$. The Eilenberg-Moore spectral sequence ([16] [5]) for the pullback diagram can be constructed under some condition on the connectivity of function spaces.

Theorem 2.1. *Suppose that* $i^\natural$ *is surjective and that* $\mathcal{F}(A, Z)$ *is a simply-connected space whose integral homology is of finite type. Assume further that* $\mathcal{F}(Y, Z)$ *is connected and has integral homology of finite type. Then there exists a spectral sequence converging to the cohomology* $H^*(\mathcal{F}(Y \cup_\alpha X, Z); R)$ *as an algebra with*

$$E_2^{*,*} \cong \mathrm{Tor}_{H^*(\mathcal{F}(A,Z);R)}(H^*(\mathcal{F}(X, Z); R), H^*(\mathcal{F}(Y, Z); R))$$

as a bigraded algebra.

Observe that Theorem 2.1 remains valid even if the functor $\mathcal{F}(\ ,Z)$ is replaced with $\mathcal{F}_*(\ ,Z)$ in the category of based topological spaces. Let $\Lambda^m Z$ and $\Omega^m Z$ denote the m-fold free loop space $\mathcal{F}(S^m, Z)$ and the iterated loop space $\mathcal{F}_*(S^m, Z)$, respectively. Consider the cofibration $i : S^m \to D^{m+1}$. Since the map $\gamma : Z \to \mathcal{F}(D^{m+1}, Z)$ defined by $\gamma(z)(t) = z$ is a homotopy equivalence and hence the fibration $i^\natural : \mathcal{F}(D^{m+1}, Z) \to \Lambda^m Z$ is regarded as the map $s = i^\natural \circ \gamma : Z \to \Lambda^m Z$. Notice that $\mathcal{F}_*(D^{m+1}, Z)$ is contractible. Theorem 2.1 thus yields the following result.

Corollary 2.2. *Suppose that Z is $(m+1)$-connected and that $\mathcal{F}(Y, Z)$ is a connected space.*
(i) *If the integral homologies of $\Lambda^m Z$ and $\mathcal{F}(Y, Z)$ are of finite type, then there is a spectral sequence converging to $H^*(\mathcal{F}(Y \cup_\alpha D^{m+1}, Z); R)$ with*

$$E_2^{*,*} \cong \mathrm{Tor}_{H^*(\Lambda^m Z;R)}(H^*(Z; R), H^*(\mathcal{F}(Y, Z); R)).$$

Here the $H^*(\Lambda^m Z; R)$-module structure of $H^*(Z; R)$ is defined by the induced map $H^*(s; R)$.
(ii) *If the integral homologies of $\Omega^m Z$ and $\mathcal{F}(Y, Z)$ are of finite type, then there is a spectral sequence converging to $H^*(\mathcal{F}_*(Y \cup_\alpha D^{m+1}, Z); R)$ with*

$$E_2^{*,*} \cong \mathrm{Tor}_{H^*(\Omega^m Z;R)}(R, H^*(\mathcal{F}_*(Y, Z); R)).$$

If R is taken to be the field $\mathbb{F}_p$, then the Hopf algebra $H^*(\Omega^m Z; \mathbb{F}_p)$ is a tensor product of monogenic algebras. Therefore, it is not difficult to construct a Koszul resolution of $\mathbb{F}_p$ as an $H^*(\Omega^m Z; \mathbb{F}_p)$-module. Assume further that the images of generators of the monogenic algebras by the induced map $H^*(\alpha^\natural; \mathbb{F}_p) : H^*(\Omega^m Z; \mathbb{F}_p) \to H^*(\mathcal{F}_*(Y, Z); \mathbb{F}_p)$ are expressed exactly in terms of the elements in $H^*(\mathcal{F}_*(Y, Z); \mathbb{F}_p)$. Then the differential graded algebra (E_1, d_1), which computes the E_2-term, can be determined.

Remark 2.3. The EMSS (Corollary 2.2(i)) associated with the functor $\mathcal{F}(\ ,Z)$ and the push out diagram

$$\begin{array}{ccc} S^{m+1} & \xleftarrow{\alpha_X} & D^{m+1} \\ \uparrow & & \uparrow{\scriptstyle i} \\ pt & \longleftarrow & S^m \end{array}$$

has been used in [9] to consider the problem of determining when the evaluation fibration $\Omega^{m+1} Z \to \Lambda^{m+1} Z \to Z$ is totally non homologous to zero with respect to a given field. Observe that when $m = 0$, the EMSS is nothing but the spectral sequence associated with the homotopy fibre square (1.1) mentioned in the Introduction.

3. Sketch of the proof of Theorem 1.1

Throughout this section, let M be a closed orientable 3-dimensional manifold. Let $M = (-H_g) \cup_{rh} H_g$ be a Heegaard splitting of M, where H_g denotes the handlebody whose boundary is the Riemann surface Σ_g of genus g, $h : \Sigma_g \to \Sigma_g$ and

$r : H_g \to -H_g$ are an orientation-preserving homeomorphism and an orientation-reversing map, respectively. Thus we have a pushout diagram (3.1):

$$
\begin{array}{ccccc}
M = & (-H_g) \cup_{rh} H_g & \longleftarrow & & H_g \\
 & \uparrow & & & \uparrow \kappa \\
 & -H_g & \xleftarrow{r} & H_g \xleftarrow{\kappa} \Sigma_g \xleftarrow{h} & \Sigma_g,
\end{array}
$$

in which $\kappa : \Sigma_g \to H_g$ stands for the inclusion. Since the handlebody H_g is a 3-dimensional CW-complex and BG is 3-connected, it follows that the induced map $\kappa^\natural : \mathcal{F}(H_g, BG) \to \mathcal{F}(\Sigma_g, BG)$ is surjective. Theorem 2.1 enables us to obtain the EMSS converging to $H^*(\mathcal{F}(M, BG); R)$ with

$$E_2^{*,*} \cong \mathrm{Tor}_{H^*(\mathcal{F}(\Sigma_g,BG);R)}(H^*(\mathcal{F}(H_g, BG); R), H^*(\mathcal{F}(H_g, BG); R))$$

as a bigraded algebra. In order to compute the cohomology algebra $H^*(\mathcal{F}(\Sigma_g BG); \mathbb{F}_p)$ in low degrees, we first consider the cofibre square

$$
\begin{array}{ccc}
\vee_{i=1}^{q} S^1 & \longleftarrow & S^1 \\
\uparrow & & \uparrow i \\
\vee_{i=1}^{q-1} S^1 & \longleftarrow & pt,
\end{array}
$$

From Theorem 2.1, we see that, as an algebra,

$$H^*(\mathcal{F}(\vee_{k=1}^{2g} S^1, BG); R) \cong R[c_i] \otimes \bigotimes_{k=1}^{g} \Lambda(x_{ik}^{(l)}) \otimes \bigotimes_{k=1}^{g} \Lambda(x_{ik}^{(m)})$$

for which $ev^*(c_i) = 1 \otimes c_i + \sum_{k=1}^{g} \iota_k^{(l)*} \otimes x_{ik}^{(l)} + \sum_{k=1}^{g} \iota_k^{(m)*} \otimes x_{ik}^{(m)}$, where $ev : \vee_{k=1}^{2g} S^1 \times \mathcal{F}(\vee_{k=1}^{2g} S^1, BG) \to BG$ denotes the evaluation map. Since Σ_g is regarded as the adjunction space $\vee_{l=1}^{2g} S^1 \cup_{[\alpha_1,\beta_1]\cdots[\alpha_{2g},\beta_{2g}]} D^2$ in which α_i and β_i are generators of $\pi_1(\Sigma_g)$, it follows that the Riemann surface fits in the cofibre square

$$
\begin{array}{ccc}
\Sigma_g & \xleftarrow{j} & D^2 \\
i \uparrow & & \uparrow \iota \\
\vee_{l=1}^{2g} S^1 & \xleftarrow{\alpha} & S^1,
\end{array}
$$

where α denotes the attaching map $[\alpha_1, \beta_1] \cdots [\alpha_{2g}, \beta_{2g}]$. By computing the EMSS associated with the above cofibre square in low degrees, we have

Lemma 3.1. *For* $* \leq 5$, *as an algebra,*

$$
\begin{aligned}
H^*(\mathcal{F}(\Sigma_g, BG); R) \quad \cong \quad & R[c_i] \otimes \bigotimes_{k=1}^{g} \Lambda(x_{ik}^{(l)}) \otimes \bigotimes_{k=1}^{g} \Lambda(x_{ik}^{(m)}) \\
& \otimes \frac{R[s^{-1}x_i, \gamma_2(s^{-1}x_i)]}{\Big((s^{-1}x_i)^2 - 2\gamma_2(s^{-1}x_i) \Big)},
\end{aligned}
$$

where $\deg x_{ik}^{(l)} = \deg x_{ik}^{(m)} = \deg c_i - 1$, $\deg s^{-1}x_i = \deg c_i - 2$ *and* $\deg \gamma_2(s^{-1}x_i) = 2(\deg c_i - 2)$. *The elements* c_i, $x_{ik}^{(l)}$ *and* $x_{ik}^{(m)}$ *in* $H^*(\mathcal{F}(\Sigma_g, BG); R)$ *can be chosen so that* $(ev_*)^*(c_i) = c_i$ *and* $ev_{\Sigma_g}^*(c_i) = \sum_{k=1}^g t_k^{(l)*} \otimes x_{ik}^{(l)} + \sum_{k=1}^g t_k^{(m)*} \otimes x_{ik}^{(m)}$ *in* $\oplus_{3\le j\le 5} H^1(\Sigma_g) \otimes H^j(\mathcal{F}(\Sigma_g, BG); R)$.

The fact that the handlebody H_g is homotopy equivalent to the wedge $\vee_{k=1}^g S^1$ enables us to deduce the following lemma.

Lemma 3.2. *As an algebra,*

$$H^*(\mathcal{F}(H_g, BG); R) \cong R[c_i] \otimes \bigotimes_{k=1}^{g} \Lambda(x_{ik}).$$

Let $\mathcal{M}_g$ be the mapping class group of the surface Σ_g and let $\rho : \mathcal{M}_g \to \mathrm{GL}(2g, \mathbb{Z})$ be the canonical representation associated with the integral homology $H_1(\Sigma_g; \mathbb{Z}) = \mathbb{Z}\{t_1^{(l)}, ..., t_g^{(l)}, t_1^{(m)}, ..., t_g^{(m)}\}$. Analyzing the induced map $(h^\natural)^*$ on $H^*(\mathcal{F}(\Sigma_g, BG); R)$, we have

Lemma 3.3. *Let* $x_{ik}^{(l)}$, *and* $x_{ik}^{(m)}$ *be the indecomposable elements of the cohomology ring* $H^*(\mathcal{F}(\Sigma_g, BG); R)$ *mentioned in Lemma 3.1. The action* $(h^\natural)^*$ *on* $H^*(\mathcal{F}(\Sigma_g, BG); R)$ *is given by*

$$\begin{aligned}((h^\natural)^*(x_{i1}^{(l)}), ..., (h^\natural)^*(x_{ig}^{(l)}), (h^\natural)^*(x_{i1}^{(m)}), ..., (h^\natural)^*(x_{ig}^{(m)}))& \\ = (x_{i1}^{(l)}, ..., x_{ig}^{(l)}, x_{i1}^{(m)}, ..., x_{ig}^{(m)})\ \rho(h).&\end{aligned}$$

The following theorem is a key to proving Theorem 1.1.

Theorem 3.4. *Suppose that* $\mathrm{Tor}_{\mathbb{Z}}(\mathbb{Z}/2, R) = 0$. *Then, as a bigraded algebra,*

$$\begin{aligned}E_\infty^{*,*} \quad \cong \quad E_2^{*,*} &\cong R[c_i] \otimes \Big(\bigotimes \Lambda(x_{ik})/\mathrm{Im}\ ((r\kappa h)^\natural)^*\Big) \\ &\otimes \mathrm{Ker}((r\kappa h)^\natural)^* \otimes \Lambda(\beta_i)/(-2\beta_i) \otimes \Lambda(s^{-1}s^{-1}x_i)\end{aligned}$$

for total degree ≤ 3, *where* $\mathrm{bideg}\ c_i = (0, \deg c_i)$, $\mathrm{bideg}\ x_{ik} = (0, 3)$, $\mathrm{bideg}\ \beta_i = (-1, 2(\deg c_i - 2))$ *and* $\mathrm{bideg}\ s^{-1}s^{-1}x_i = (-1, \deg c_i - 2)$. *Here* $((r\kappa h)^\natural)^*$ *is viewed as the map from* $\bigotimes_{k=1}^g \Lambda(x_{ik}^{(m)})$ *to* $\bigotimes_{k=1}^g \Lambda(x_{ik})$. *Moreover* $\big\{\bigotimes_{k=1}^g \Lambda(x_{ik})/\mathrm{Im}\ ((r\kappa h)^\natural)^*\big\}^3$ *is isomorphic to* $H_1(M; R)^{\oplus r}$ *for which* $r = \mathrm{rank}\ H^4(BG; \mathbb{Z})$.

By constructing the Koszul-Tate resolution of the cohomology $H^*(\mathcal{F}(H_g, BG); \mathbb{F}_p)$ as a $H^*(\mathcal{F}(\Sigma_g, BG); \mathbb{F}_p)$-module by hand, we see that $E_2^{-2,4} = \mathrm{Tor}_{\mathbb{Z}}(\mathbb{Z}/2, R)^{\oplus r}$ and hence $E_2^{-2,4} = 0$ by assumption. Moreover it follows from the explicit calculation of the E_2-term that there is no algebra generator in the area $E_2^{i,j}$ for $i \le -2$ and $i + j \le 3$ (see Figure (3.2)). Thus we can get the first half of Theorem 3.4.

Consider the following sequence of maps

$$\pi_1(H_g) = \langle y_1, ..., y_g \mid \quad \rangle \xleftarrow{h_*} \pi_*(\Sigma_g) \xleftarrow{\vee i^{(m)}} \pi_1(\vee^g S^1) = \langle t_1^{(m)}, ..., t_g^{(m)} \mid \quad \rangle.$$

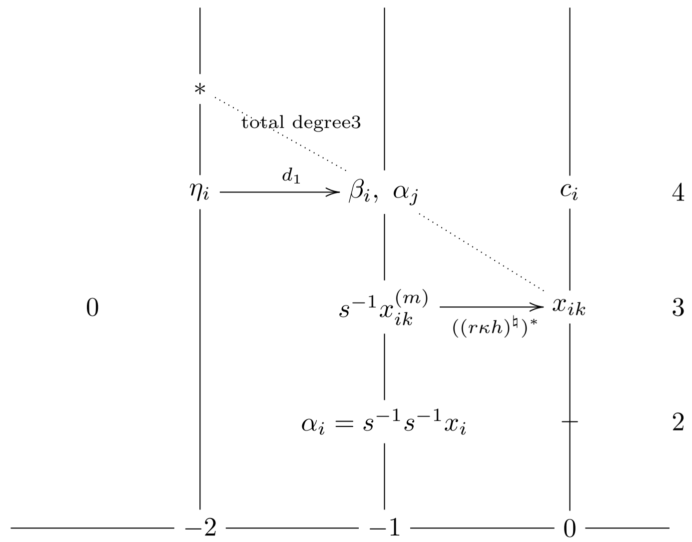
As is well known $\pi_1(M) = \langle y_1, ..., y_g \mid h_*(t_1^{(m)}), ..., h_*(t_g^{(m)})\rangle$. Put $\rho(h) = \big(a_{st}\big) = \begin{pmatrix} * & A \\ * & * \end{pmatrix}$, where $A \in \mathrm{GL}(g, \mathbb{Z})$. Then we have $H_1(M; \mathbb{Z}) = \pi_1(M)/[\pi_1(M), \pi_1(M)] =$

$\mathbb{Z}^{\oplus g}/\mathrm{Im}\,A$. Therefore we can conclude that $\mathbb{Z}^{\oplus g}\otimes R/\mathrm{Im}\,(A\otimes 1) = (\mathbb{Z}^{\oplus g}/\mathrm{Im}\,A)\otimes R = H_1(M;\mathbb{Z})\otimes R = H_1(M;R)$. It follows from Lemma 3.3 that

$$((r\kappa h)^{\natural})^*(x_{ik}^{(m)}) = r^{\natural *}\kappa^{\natural *}h^{\natural *}x_{ik}^{(m)} = \sum_{1\le t\le g} a_{t\ g+k}x_{it}.$$

This fact allows us to deduce the latter half of Theorem 3.4.

Figure (3.2)



In general, it is not easy to solve extension problems which appears in the E_∞-term of the EMSS. Fortunately the problem can be solved when the source space is a homology 3-sphere.

Proposition 3.5. *Let R be an integral domain and M a homology 3-sphere. Assume that $H^*(BG;R)\cong R[y_{4,i},y_{6,j},..]$ as an algebra. Then, for $*\le 3$,*

$$\begin{aligned} H^*(\mathcal{F}(M,BG);R) &\cong H^2\otimes R\{\beta_i\}/(2\beta_i)^{\oplus r} \\ &\qquad \otimes\Lambda(s^{-1}s^{-1}x_{6,j})\otimes\Lambda(s^{-1}s^{-1}x_{4,j}) \end{aligned}$$

as an algebra, where rank $H^4(BG;\mathbb{Z}) = r$, $H^2 = H^2(\mathcal{F}(M,BG);R) = \mathrm{Tor}_{\mathbb{Z}}(\mathbb{Z}/2,R)^{\oplus r}$, $\deg\beta_i = \deg x_{4,i} = \deg s^{-1}s^{-1}x_{6,j} = 3$ *and* $\deg s^{-1}s^{-1}x_{4,j} = 1$.

After the author's talk in the conference Trends in Topology, Professor Kono has suggested a more smart way to show the appearance of the first homology of a closed 3-dimensional manifold M in the cohomology of the function space $\mathcal{F}(M,BG)$. The author hopes that the study of the function space is made via the way in a forthcoming paper.

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Department of Applied Mathematics, Faculty of Science, Okayama University of Science, Okayama 700-0005, Japan `kuri@geom.xmath.ous.ac.jp`