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TOWARDS MIRROR SYMMETRY OF NONABELIAN QUOTIENTS

BUMSIG KIM

1. Symplectic Reduction

Let (X, ω) be a symplectic manifold with a **Hamiltonian** group action by a compact Lie group G. Then by the very definition there is a G-equivariant differentiable map $\mu : X \to \mathcal{G}^*$ such that

$$X_v \rfloor \omega = d\mu_v$$

where $v \in \mathcal{G}$ (a Lie algebra element of G) and $\mu_v(x) := (\mu(x))(v)$, $x \in X$, and X_v is the vector field on X generated by v. The map μ is called a **moment map**.

Now choose a regular point of the moment map μ, say, $0 \in \mathcal{G}^*$. Then a **symplectic reduction** (with respect to the regular point 0) is by definition

$$X//G := \mu^{-1}(0)/G.$$

Assume that G action on $\mu^{-1}(0)$ is a free action and $\mu^{-1}(0)/G$ is compact.

Example: Consider $\mathbb{C}^{n+1}$ as a symplectic vector space with the standard symplectic form. Let S^1 act on $\mathbb{C}^{n+1}$ by $t(z_0, ..., z_n) = (tz_0, ..., tz_n)$ where $S^1 = \{t \in \mathbb{C}| \ |t| = 1\}$. Then $\mu : \mathbb{C}^{n+1} \to \mathbb{R}$ defined by $\mu(z) = \sum |z_i|^2 + 1$ is a moment map for the action. So, $(\mu^{-1}(0) = S^{2n+1})/S^1 = \mathbb{C}P^n$.

Based on mirror symmetry I speculate that **the geometry of $X//G$ is describable by the geometry of $X//T$ where T is a maximal torus of G.** Here the geometry means the derived category of coherent sheaves, K - groups, cohomology, Lagrangian intersection theory, enumerative algebraic geometry, and so on. Note that if $\pi : \mathcal{G}^* \to \mathfrak{t}^*$ is the natural projection, then $\nu = \pi \circ \mu$ is a moment map for the T-action. So we let here $X//T := \nu^{-1}(0)/T$. There are some evidences for this wild speculation.

Theorem *(Ellingsrud - Strømme (*[ES] *when X is a symplectic vector spaces, 1989), Matin (*[Mar] *when X is a symplectic manifold, 2000))*

$$H^*(X//G, \mathbb{Q}) \cong \frac{H^*(X//T, \mathbb{Q})^W}{\mathrm{ann}(e)}$$

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where

$$e = \prod_{\alpha \text{ all roots}} c_1(L_\alpha),$$

$L_\alpha = (\mu^{-1}(0) \times \mathbb{C})/T$, *$W$ is the Weyl group of (G,T), and* $\text{ann}(e) = \{a \in H^*(X//T)^W \mid a \cup e = 0\}$.

The following diagram is a basic picture for the comparison.

$$\begin{array}{ccc} \nu^{-1}(0)/T & \xleftarrow[\supset\, i]{} & \mu^{-1}(0)/T \\ & & \Big\downarrow \pi \\ & & \mu^{-1}(0)/G. \end{array}$$

There is **Martin's integration formula**:

$$\int_{X//G} a = \frac{1}{|W|} \int_{X//T} \tilde{a} \cup e$$

where $\tilde{a}$ is a lift of a, i.e., $i^*(\tilde{a}) = \pi^*(a)$. This can be seen easily from the diagram after analyzing the relative tangent bundle and the normal bundle appeared in the basic picture. Now using Borel's theorem and Kirwan's map one can prove the theorem.

Example Let $r < n$. $U(r)$ acts on $X = \text{Mat}_{r\times n}(\mathbb{C})$. Then one may take $\mu(A) = AA^* - I$ and $\nu(A) =$ diagonal part of $(AA^* - I)$ as moment maps. Thus $\mu^{-1}(0) = \{r \text{ unitaric vectors } \mathbb{C}^n\}$ and $\nu^{-1}(0) = \{r \text{ unit vectors in } \mathbb{C}^n\}$. Hence

$X//G = \mu^{-1}(0)/U(r) = \{\ r\text{-dim subspace in } \mathbb{C}^n\ \}$
$\quad = G(r,n) =$ Grassmannian

$X//T = \{\ r\text{-many 1-dim subspace in } \mathbb{C}^n\ \}$
$\quad = (\mathbb{P}^{n-1})^r$

Note that $W \cong S_r \circlearrowleft T = (S^1)^r$ and $H^*(\mathbb{C}P^{n-1})^r = \mathbb{Z}[\chi_1,...,\chi_r]/(\chi_1^n,...,\chi_r^n)$ where $\chi_i = c_1(\mathcal{O}(1))$. By the theorem we conclude that

$H^*G(r,n) \cong \mathbb{Z}[c_1,...,c_r]/(Y_{n-r+1},...,Y_n) \cong \mathbb{Z}[\chi_1,...,\chi_r]^W/(\chi_1^n,...,\chi_r^n : \prod_{i\neq j}(\chi_i - \chi_j))$ where $c_i = c_i(S^*)$=i-th elementary symetic. poly. in χ_i and Y_j are the j-th Segre class of S^*. (By definition, $c(S^*)(1 + Y_1 + Y_2 + ...) = 1$.)

2. Is there an analogous relationship between quantum cohomology $QH^*(X//T)$ and $QH^*(X//G)$?

Let (X,ω) be a compact Kähler manifold (or in general, a compact symplectic manifold). Let's make use of data $f : S^2 = \mathbb{C}P^1 \to X$ where f is (pseudo-)holomorphic in order to construct a new ring structure on

$$H^*(X) \otimes \mathbb{Q}[[e^t]]$$

(with respect to a compatible almost complex structure on X).

For simple presentation, assume that

$$H^{\text{even}}(X) = H^*(X).$$

The rational 3-pointed correlators $< a, b, c >$ define a new ring structure on the vector space $QH^*(X) := H^*(X) \otimes \mathbb{Q}[[e^t]]$:

$$a * b = \sum_{i,j} < a, b, \phi_i > g^{ij} \phi_j$$

where ϕ_i is a basis of $H^*(X) = H^{\text{even}}(X)$, $g_{ij} = \int_X \phi_i \cup \phi_j$, and $(g^{ij}) = (g_{ij})^{-1}$.

Naively, define, for $d \in H_2(X, \mathbb{Z})$
$< A_1, A_2, ..., A_m >_{g,d,X} :=$ the number of holomorphic maps $(\Sigma_g, x_1, ..., x_m) \to X$ such that $f(x_i) \in A_i$ and $[f(\Sigma_g)] = d$.

Define

$$< A, B, C >= \sum_{d \in H_2(X,\mathbb{Z})} e^{<\omega,d>t} < A, B, C >_{0,d,X}$$

Example Let $\mathbb{C}P^n = \mathbf{P}^n$ with the symplectic form $\omega := c_1(\mathcal{O}(1))$. Denote the hyperplane class in $\mathbf{P}^n$ by H. Then $H^n =$ a point class.

Note that a $\mathbb{Q}[e^t]$-basis of $QH^*(\mathbf{P}^n)$ is $H^0 = 1$, H, H^2, ...,H^n.

$$< H, H^n, H^n >_{0,d,\mathbf{P}^n} = \begin{cases} 1 & \text{if } d = 1 \\ 0 & \text{otherwise} \end{cases}$$

So $H * H^n = e^t H^0$.

The naive definition works if X is a simply connected homogeneous Kähler manifolds and $g = 0$. In general for a precise definition of the correlators, one needs highly nontrivial works by some people.

Theorem *(Bertram, Ciocan-Fontanine, Kim* [BCK]*)*

$$< a, b, c >_{G(r,n)} = \frac{(-1)^{r(r-1)/2}}{r!} < a\sqrt{e}, b\sqrt{e}, c >_{(\mathbf{P}^{n-1})^r}$$

where $\omega_{G(r,n)} = c_1(S^*)$, $\omega_{(\mathbf{P}^{n-1})^r} = \sum_{i=1}^r c_1(\pi_i^* \mathcal{O}_{\mathbf{P}^{n-1}}(1))$, $t_{G(r,n)} + (r-1)\pi\sqrt{-1} = t_{(\mathbf{P}^{n-1})^r}$, $\sqrt{e} = \prod_{r \geq i > j \geq 1} (\chi_i - \chi_j) =$ *the fundamental anti-invariant where* $\chi_i = c_1(\pi_i(\mathcal{O}_{\mathbf{P}^{n-1}}(1)))$ *and* $\pi : (\mathbf{P}^{n-1})^r \to \mathbf{P}^{n-1}$.

Corollary

$$QH^*(G(r,n)) = \frac{\mathbb{Q}[\chi_1, ..., \chi_r]^W \otimes \mathbb{Q}[e^t]}{(\chi^n + (-1)^r e^t, ..., \chi^n + (-1)^r e^t : e)}$$

There are two applications for the result.

- Mirror Symmetry for Grassmannians (= the Hori - Vafa conjecture [HV]) holds.
- The Virasoro conjecture Grassmannians is true, which is a structural theorem for all genus Gromov - Witten Invariants.

Now it is natural to ask **open questions**.

Prove the comparison formula: for $\gamma_i \in H^*(X//G)$,

$$\begin{aligned} &< \gamma_1, \gamma_2, ..., \gamma_m >_{0,d,X//G} \\ = \; & \frac{\pm}{|W|} \sum_{\mathbf{d} \mapsto d} < \sqrt{e}\tilde{\gamma}_1, \sqrt{e}\tilde{\gamma}_2, \tilde{\gamma}_3, ... >_{0,\mathbf{d},X//T} \end{aligned}$$

for general case X and for any m-pointed correlators.

Remark So far, when $X//G =$ partial flag manifolds, or symplectic flag manifolds (and so on) the comparison formula are proven for $m = 3$ case only.

3. Mirror symmetry conjecture

Roughly speaking, the mirror conjecture is as follows.

When X is (a certain class of) compact Calabi-Yau (i.e., $c_1(T_X) = 0$), the Frobenius structure $QH^*(X)$ (nonlinear σ-model) is conjecturally isomorphic to the Frobenius structure of $\frac{\infty}{2}$ Hodge variations of another Calabi-Yau Y (Quantum Periods).

When X is (a certain class of) Fano (i.e, $c_1(T_X)$ is a Kähler form), the Frobenius structure $QH^*(X)$ (nonlinear σ model) is conjecturally isomorphic to the Saito's flat structure (V, f, Ω) (Landau-Ginzburg model) where V is a smooth affine variety, $f : V \to \mathbb{C}$ is an algebraic function (=superpotential) with finite, nondegenerated critical points, and Ω is a holomorphic top form.

In particular, $A_{ij}^k \leftrightarrow B_{ij}^k$ where B is defined by

$$\hbar \partial_{t_i} \partial_{t_j} \int_\Gamma e^{F/\hbar} \Omega = \sum_k B_{ij}^k \partial_{t_k} \int_\Gamma e^{F/\hbar} \Omega$$

where F is the universal unfolding of f, $\Gamma \in H_n(Y; \mathrm{Re}(F/\hbar) = -\infty)$, and t are suitable parameters for universal unfolding. Or the oscillatory integrals $\int_\Gamma e^{F/\hbar}\Omega$ are **flat coordinates** of the extended connection of the Gauss-Manin connection.

Theorem *(Givental* [G]*, Barannikov* [B]*) The mirror partner of* $\mathbb{C}P^n$ *is*

$$((\mathbb{C}^*)^n, \; \sum_{i=1}^n x_i + \frac{1}{x_1 ... x_n}, \; \bigwedge_{i=1}^n \frac{dx_i}{x_i})$$

Givental proved the above theorem restricted to the small parameter and Barannikov proved the theorem fully. Mirror partners of Calabi-Yau or Fano complete intersections in toric manifold are known by Batyrev - Borisov and are verified in small QH^* sense by Givental. Mirror partners of flag manifolds are known to be generalized Whittaker functions expressed in oscillatory integrals and verified in small sense by Givental - Kim.

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School of Math. KIAS 207-43 Cheongnyangni 2-dong, Dongdaemun-gu, Seoul, 130-722 Korea