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INFRA-NILMANIFOLD WITH QUATERNIONIC NIL-GEOMETRY

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ABSTRACT. we shall study the quaternionic Heisenberg group $\mathcal{H}_7(\mathbb{H}) = \mathbb{R}^3 \tilde{\times} \mathbb{R}^4$ which is important to investigate an *almost Bieberbach group* of a 7-dimensinal infra-nilmanifold.

1. Introduction

Let L be a connected and simply connected nilpotent Lie group. Then $\mathrm{Aff}(L) = L \rtimes \mathrm{Aut}(L)$ is called the affine group of L, where the group operation is given by

$$(g, \alpha)(h, \beta) = (g \cdot \alpha(h), \alpha\beta)$$

and $\mathrm{Aff}(L)$ acts on L by $(g, \alpha)x = g \cdot \alpha(x)$ for $(g, \alpha) \in \mathrm{Aff}(L)$ and $x \in L$. For example, if $L = \mathbb{R}^n$, the group operation is given by

$$(a, A)(b, B) = (a + Ab, AB)$$

and $\mathrm{Aff}(n) = \mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{R})$ acts on $\mathbb{R}^n$ by

$$(a, A)x = a + Ax$$

for (a, A), $(b, B) \in \mathrm{Aff}(n)$ and $x \in \mathbb{R}^n$.

Let K be any maximal compact subgroup of $\mathrm{Aut}(L)$. Then a discrete uniform subgroup E of $L \rtimes K$ is called an *almost crystallographic group*. E is a torsion-free if and only if the E-action on L is free. In this case E is called an *almost Bieberbach group* (=AB-group) and the coset space $E\backslash L$ is an infra-nilmanifold. (In case $E \subset L$, $E\backslash L$ is called a nilmanifold.) If L is abelian ($\cong \mathbb{R}^n$ for some n), this terminology reduces to a *crystallographic group*, a *Bieberbach group* and a *flat Riemannian manifold*, respectively. Almost Bieberbach groups are exactly the fundamental groups of compact infra-nilmanifolds. A closed 3–dimensional manifold has a Nil–geometry if it is an infra–nilmanifold. It is well known that infra-nilmanifolds are determined completely (up to affine diffeomorphism) by their fundamental groups E; every 2–step infra-nilmanifold has an affine structure.

The complex Heisenberg group $\mathbf{H}_{2n+1}(\mathbb{C})$ is

$$\mathbf{H}_{2n+1}(\mathbb{C}) = \mathbb{R} \tilde{\times} \mathbb{C}^n$$

with group operation given by

$$(s, \mathbf{z})(t, \mathbf{z}') = (s + t + 2\mathrm{Im}\{\mathbf{z}\overline{\mathbf{z}}'\},\ \mathbf{z} + \mathbf{z}'),$$

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where $\mathrm{Im}(\mathbf{z}\overline{\mathbf{z}}')$ is the imaginary part of the complex number,

$$z_1\overline{z}_1' + z_2\overline{z}_2' + \cdots + z_n\overline{z}_n'$$

for $\mathbf{z} = (z_1, z_2, \cdots, z_n)$ and $\mathbf{z}' = (z_1', z_2', \cdots, z_n')$. Then $\mathbf{H}_{2n+1}(\mathbb{C})$ is a simply connected 2-step nilpotent Lie group with the center $\mathcal{Z}(\mathbf{H}_{2n+1}(\mathbb{C})) = \mathbb{R}$. Let M be an infra-nilmanifold with $\mathbf{H}_{2n+1}(\mathbb{C})$-geometry; that is, $M = \Pi\backslash\mathbf{H}_{2n+1}(\mathbb{C})$, where Π is a torsion free, discrete, cocompact subgroup of $\mathbf{H}_{2n+1}(\mathbb{C}) \rtimes C$ for some compact subgroup C of $\mathrm{Aut}(\mathbf{H}_{2n+1}(\mathbb{C}))$. Such a group Π is called an AB-group. It is well known (see for example [3]) that Π contains a cocompact lattice of $\mathbf{H}_{2n+1}(\mathbb{C})$ with index bounded above by a universal constant I. That is, I is the maximal order of the holonomy groups. Note that the analogue in the Euclidean case is a consequence of a theorem of Bieberbach which states that there are only finitely many flat manifolds in each dimension. It is shown in [8] that when $n = 2$, $I = 24$. As a consequence ([8, Corollary]), the minimum volume of a complex hyperbolic 3-manifold M of finite volume with k cusps is $k/9$, and k is at most $-24\pi^3\chi(M)$.

The quaternionic Heisenberg group $\mathbf{H}_{4n+3}(\mathbb{H})$ is

$$\mathbf{H}_{4n+3}(\mathbb{H}) = \mathbb{R}^3 \tilde{\times} \mathbb{H}^n$$

with group operation given by

$$(s, \mathbf{q})(t, \mathbf{q}') = (s + t + 2\mathrm{Im}\{\mathbf{q}\overline{\mathbf{q}}'\}, \mathbf{q} + \mathbf{q}'),$$

for $\mathbf{q} = (q_1, q_2, \cdots, q_n)$, $\mathbf{q}' = (q_1', q_2', \cdots, q_n') \in \mathbb{H}^n$, where

$$q_i = x_1 + ix_2 + jx_3 + kx_4, \qquad \bar{q}_i = x_1 - ix_2 - jx_3 - kx_4.$$

Note that $\mathrm{Im}\{\mathbf{q}\overline{\mathbf{q}}'\}$ is the imaginary part of the quaternion number,

$$q_1\overline{q_1}' + q_2\overline{q_2}' + \cdots + q_n\overline{q_n}'$$

seen as an element of $\mathbb{R}^3$. Then $\mathbf{H}_{4n+3}(\mathbb{H})$ is a simply connected 2-step nilpotent Lie group with the center $\mathcal{Z}(\mathbf{H}_{4n+3}(\mathbb{H})) = \mathbb{R}^3$.

Let M be an infra-nilmanifold with $\mathbf{H}_{4n+3}(\mathbb{H})$-geometry; that is, $M = \Pi\backslash\mathbf{H}_{4n+3}(\mathbb{H})$, where $\Pi \subset \mathbf{H}_{4n+3}(\mathbb{H}) \rtimes C$ is a torsion free, discrete subgroup with compact quotient for a compact subgroup C of $\mathrm{Aut}(\mathbf{H}_{4n+3}(\mathbb{H}))$. Since $\Pi \cap \mathcal{Z}(\mathbf{H}_{4n+3}(\mathbb{H})) \cong \mathbb{Z}^3$ is a lattice of $\mathcal{Z}(\mathbf{H}_{4n+3}(\mathbb{H}))$, M fits

$$T^3 \to M \to N,$$

a Seifert 3-torus "bundle" over a $4n$-dimensional flat orbifold. When there is no singular point, it is a genuine bundle over the base space N which is a flat Riemannian $4n$-manifold.

Let I_{n+1} denote the maximal order of the holonomy groups of all infra-nilmanifolds with $\mathbf{H}_{4n+3}(\mathbb{H})$-geometry. Recently it is proved [2] that when $n = 1$, $I_2 = 48$. The number I_2 is significant in its own right, but here is another application. According to a recent work of Kim and Parker [4, Corollary 5.3], it is related to the minimum volume of quaternionic hyperbolic orbifolds. More precisely, let M be an $(n+1)$-dimensional quaternionic hyperbolic orbifold with k cusps. Then the volume of M

is

$$\mathrm{vol}(M) \geq \frac{2^n k}{3^n(2n+3)m\sqrt{2}},$$

where m is the maximal index of a lattice in any of the subgroups of $\pi_1(M)$ stabilizing a cusp. When $n=1$, it is shown in [4, Proposition 5.8] that $m=576$. We note that this result is still true for manifolds, and in this case $m = I_{n+1}$. Hence we have a very sharp result for manifolds; namely, the minimum volume of a 2-dimensional quaternionic hyperbolic manifold with k cusps is $\dfrac{\sqrt{2}k}{15 I_2} = \dfrac{\sqrt{2}k}{720}$.

Let $\mathfrak{so}(n)$ be the group of skew-symmetric matrices. This is the Lie algebra of the special orthogonal group $\mathrm{SO}(n)$. There is a bilinear map

$$\mathcal{I} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathfrak{so}(n)$$

defined as follows: For

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \ \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix},$$

$$\mathcal{I}(\mathbf{x}, \mathbf{y}) = \mathbf{x}\mathbf{y}^t - \mathbf{y}\mathbf{x}^t,$$

where $()^t$ denotes transpose of the matrix (). With this $\mathcal{I}$, the set $\mathfrak{so}(n) \times \mathbb{R}^n$ gets a group structure given by

$$(s, \mathbf{x})(t, \mathbf{y}) = (s + t + \mathcal{I}(\mathbf{x}, \mathbf{y}),\ \mathbf{x} + \mathbf{y}).$$

This group is denoted by

$$\mathfrak{R}^n = \mathfrak{so}(n) \widetilde{\times} \mathbb{R}^n.$$

Then $\mathfrak{R}^n$ is a simply connected nilpotent Lie group, of nilpotency class 2, with the center $\mathfrak{so}(n)$. Note that $\mathfrak{so}(n)$ is viewed as a commutative Lie group isomorphic to $\mathbb{R}^{\frac{n(n-1)}{2}}$ (not as a Lie algebra). It fits the short exact sequences of Lie groups

$$0 \to \mathfrak{so}(n) \to \mathfrak{R}^n \to \mathbb{R}^n \to 1$$

Note that for the case of $n = 2$, $\mathfrak{R}^2 = \mathfrak{so}(2) \times \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}^2$ has group law

$$(s, \mathbf{x})(t, \mathbf{y}) = (s + t + x_1 y_2 - x_2 y_1, \mathbf{x} + \mathbf{y}).$$

There is an isomorphism from the standard 3-dimensional Heisenberg group to our $\mathfrak{R}^2$ by

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mapsto \left(z - \frac{1}{2}xy, \frac{1}{\sqrt{2}} \begin{bmatrix} x \\ y \end{bmatrix} \right).$$

Let M be an infra-nilmanifold with $\mathfrak{R}^n$-geometry; that is, $M = \Pi \backslash \mathfrak{R}^n$, where Π is a torsion free, discrete, cocompact subgroup of $\mathfrak{R}^n \rtimes C$ for some compact subgroup C of $\mathrm{Aut}(\mathfrak{R}^n)$. It is well known that Π contains a cocompact lattice Γ of $\mathfrak{R}^n$ of finite index, and the quotient group Π/Γ is called the holonomy group of M.

For the case of infra-nilmanifolds modelled on the 6-dimensional nilpotent Lie group

$$\mathfrak{N}^3 = \mathfrak{so}(3)\tilde{\times}\mathbb{R}^3,$$

Lee and Shin [6] prove that the holonomy group of maximal order is $D_8 \times \mathbb{Z}_2$, of order 16.

For $\mathbf{q} = x_1 + ix_2 + jx_3 + kx_4 \in \mathbb{H}$, write

$$\mathbf{q}(1) = x_1, \quad \mathbf{q}(i) = x_2, \quad \mathbf{q}(j) = x_3, \quad \mathbf{q}(k) = x_4.$$

For $\mathbf{q}, \mathbf{q}' \in \mathbb{H}$, we define $\mathbf{q} \odot \mathbf{q}'$ as follows:

$$\mathbf{q} \odot \mathbf{q}' = (\bar{\mathbf{q}}\mathbf{q}')(1) + (\bar{\mathbf{q}}\mathbf{q}')(i) + (\mathbf{q}\bar{\mathbf{q}}')(j) + (\mathbf{q}\bar{\mathbf{q}}')(k).$$

The main concern of this paper is the quaternionic Heisenberg group $\mathcal{H}_{4n+3}(\mathbb{H})$ with pseudo-Riemannian metrics :

$$\mathcal{H}_{4n+3}(\mathbb{H}) = \mathbb{R}^3\tilde{\times}\mathbb{H}^n$$

with group operation given by

$$(s, \mathbf{q})(t, \mathbf{q}') = (s + t + 2\mathrm{Im}\{\mathbf{q} \odot \mathbf{q}'\}, \mathbf{q} + \mathbf{q}'),$$

for $\mathbf{q} = (q_1, q_2, \cdots, q_n)$, $\mathbf{q}' = (q_1', q_2', \cdots, q_n') \in \mathbb{H}^n$, where $\mathrm{Im}\{\mathbf{q} \odot \mathbf{q}'\}$ is the imaginary part of the quaternion number

$$q_1 \odot q_1{}' + q_2 \odot q_2{}' + \cdots + q_n \odot q_n{}'$$

seen as an element of $\mathbb{R}^3$. Then $\mathcal{H}_{4n+3}(\mathbb{H})$ is a simply connected 2-step nilpotent Lie group with the center $\mathcal{Z}(\mathcal{H}_{4n+3}(\mathbb{H})) = \mathbb{R}^3$. Let M be an infra-nilmanifold with $\mathcal{H}_{4n+3}(\mathbb{H})$-geometry; that is, $M = \Pi\backslash\mathcal{H}_{4n+3}(\mathbb{H})$, where $\Pi \subset \mathbf{H}_{4n+3}(\mathbb{H}) \rtimes C$ is a torsion free, discrete subgroup with compact quotient, where C is a compact subgroup of $\mathrm{Aut}(\mathcal{H}_{4n+3}(\mathbb{H}))$.

2. The Quaternionic Heisenberg Group $\mathcal{H}_7(\mathbb{H})$

From now on, we shall use $\mathcal{H}_7(\mathbb{H}) = \mathbb{R}^3\tilde{\times}\mathbb{R}^4$ rather than $\mathbb{R}^3\tilde{\times}\mathbb{H}$. We identify $\mathbb{R}^3\tilde{\times}\mathbb{H}$ with $\mathbb{R}^3\tilde{\times}\mathbb{R}^4$ by

$$(s, \mathbf{q} = x_1 + ix_2 + jx_3 + kx_4) \longleftrightarrow (s, \mathbf{x} = [x_1, x_2, x_3, x_4]^t).$$

Accordingly, we introduce a new notation for $\mathrm{Im}\{\mathbf{q} \odot \mathbf{q}'\}$. For

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix},$$

define

$$\begin{aligned} &\mathcal{I}(\mathbf{x}, \mathbf{y}) \\ &= \left(\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} - \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix}, -\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} - \begin{vmatrix} x_4 & y_4 \\ x_2 & y_2 \end{vmatrix}, -\begin{vmatrix} x_1 & y_1 \\ x_4 & y_4 \end{vmatrix} - \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} \right)^t \\ &= (\mathbf{x}^t J_1 \mathbf{y}, \mathbf{x}^t J_2 \mathbf{y}, \mathbf{x}^t J_3 \mathbf{y})^t, \end{aligned}$$

where $()^t$ denotes the transpose of a matrix, and

$$J_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; J_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \; J_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly $\mathcal{I}(\mathbf{x},\mathbf{y}) = -\mathcal{I}(\mathbf{y},\mathbf{x})$ and $\mathcal{I}(\mathbf{x},\mathbf{y})$ corresponds to $\mathrm{Im}\{\mathbf{q} \odot \mathbf{q}'\}$. Thus the group operation in $\mathbb{R}^3 \tilde{\times} \mathbb{R}^4$ becomes

$$(s,\mathbf{x})(t,\mathbf{y}) = (s + t + 2\mathcal{I}(\mathbf{x},\mathbf{y}), \mathbf{x} + \mathbf{y}).$$

Since $\mathcal{I}(\mathbf{x}, \pm\mathbf{x}) = 0$, we see easily that

$$(s,\mathbf{x})^{-1} = (-s, -\mathbf{x}).$$

Thus we have

$$[(s,\mathbf{x}),(t,\mathbf{y})] = (s,\mathbf{x})^{-1}(t,\mathbf{y})^{-1}(s,\mathbf{x})(t,\mathbf{y}) = (4\mathcal{I}(\mathbf{x},\mathbf{y}), \mathbf{0}).$$

Let

$$\sigma = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$K_i = \sigma^{-1} J_i \sigma, \quad i = 1, 2, 3.$$

Then J_1, J_2, J_3 together with K_1, K_2, K_3 form a linear basis for the vector space $\mathfrak{so}(4)$ of the skew-symmetric matrices. Therefore,

$$\mathfrak{so}(4) = \langle J_1,\ J_2,\ J_3 \rangle \oplus \langle K_1, K_2, K_3 \rangle$$

as vector spaces. Notice that each subspace fails to be a Lie subalgebra.

In order to understand the automorphism group of $\mathcal{H}_7(\mathbb{H})$, we need to study more general setting: For any $C \in \mathrm{GL}(4,\mathbb{R})$ and $V \in \mathfrak{so}(4)$,

$$J_C(V) = C^t V C$$

defines a linear isomorphism $J_C : \mathfrak{so}(4) \longrightarrow \mathfrak{so}(4)$. In fact, with respect to the basis $\{J_1, J_2, J_3, K_1, K_2, K_3\}$, it turns out that $\det(J_C) = (\det(C))^3$.

Define

$$O(\mathbf{J}; 2, 2) = \{C \in \mathrm{GL}(4,\mathbb{R}) \mid C^t J_i C \in \langle J_1,\ J_2,\ J_3 \rangle\}.$$

That is, $C \in O(\mathbf{J}; 2, 2)$ if and only if the map J_C leaves the subspace spanned by J_1, J_2, J_3 invariant. Therefore,

$$C^t J_i C = \lambda_{i1} J_1 + \lambda_{i2} J_2 + \lambda_{i3} J_3, \quad \lambda_{ij} \in \mathbb{R},$$

for $i = 1, 2, 3$. It turns out then, the matrix $\lambda = (\lambda_{ij})$ is non-singular.

Now we form the column vector

$$\mathbf{J} = \begin{bmatrix} J_1 \\ J_2 \\ J_3 \end{bmatrix}$$

with entries the matrices J_1, J_2, J_3. With some abuse of notation, we can write

$$O(\mathbf{J}; 2, 2) = \{C \in \mathrm{GL}(4, \mathbb{R}) \mid C^t \mathbf{J} C = \lambda \mathbf{J}, \lambda \in \mathrm{GL}(3, \mathbb{R})\}.$$

Clearly $O(\mathbf{J}; 2, 2)$ is a closed subgroup of $\mathrm{GL}(4, \mathbb{R})$. For $C \in O(\mathbf{J}; 2, 2)$, let $\widehat{C} \in \mathrm{GL}(3, \mathbb{R})$ denote the nonsingular 3×3 matrix λ which satisfies $C^t \mathbf{J} C = \lambda \mathbf{J}$. So,

$$C^t \mathbf{J} C = \widehat{C} \mathbf{J}.$$

Then $C \mapsto \widehat{C}$ defines a homomorphism $\widehat{\ } : O(\mathbf{J}; 2, 2) \to \mathrm{GL}(3, \mathbb{R})$. As a map $J_C : \mathfrak{so}(4) \longrightarrow \mathfrak{so}(4)$, with respect to the ordered basis $\{J_1, J_2, J_3, K_1, K_2, K_3\}$, J_C has a matrix

$$J_C = \begin{bmatrix} \widehat{C} & 0 \\ 0 & * \end{bmatrix}.$$

Since the center, $\mathcal{Z}(\mathbb{R}^3 \tilde{\times} \mathbb{R}^4) = \mathbb{R}^3$, is a characteristic subgroup of $\mathbb{R}^3 \tilde{\times} \mathbb{R}^4$, every automorphism of $\mathbb{R}^3 \tilde{\times} \mathbb{R}^4$ restricts to an automorphism of $\mathbb{R}^3$. Consequently an automorphism of $\mathbb{R}^3 \tilde{\times} \mathbb{R}^4$ induces an automorphism on the quotient group $\mathbb{R}^4$. Thus there is a natural homomorphism

$$\mathrm{Aut}(\mathbb{R}^3 \tilde{\times} \mathbb{R}^4) \to \mathrm{Aut}(\mathbb{R}^3) \times \mathrm{Aut}(\mathbb{R}^4), \quad \theta \mapsto (\hat{\theta}, \bar{\theta}).$$

Lemma 2.1. $\mathrm{Image}\{\mathrm{Aut}(\mathbb{R}^3 \tilde{\times} \mathbb{R}^4) \to \mathrm{Aut}(\mathbb{R}^4)\} = O(\mathbf{J}; 2, 2)$. *Moreover, the exact sequence* $\mathrm{Aut}(\mathbb{R}^3 \tilde{\times} \mathbb{R}^4) \to O(\mathbf{J}; 2, 2) \to 1$ *splits.*

Proof. Let $\theta \in \mathrm{Aut}(\mathbb{R}^3 \tilde{\times} \mathbb{R}^4)$. Then $(\hat{\theta}, \bar{\theta}) \in \mathrm{Aut}(\mathbb{R}^3) \times \mathrm{Aut}(\mathbb{R}^4)$. Since $[(s, \mathbf{x}), (t, \mathbf{y})] = (4\mathcal{I}(\mathbf{x}, \mathbf{y}), \mathbf{0})$,

$$\theta[(s, \mathbf{x}), (t, \mathbf{y})] = \theta(4\mathcal{I}(\mathbf{x}, \mathbf{y}), \mathbf{0}) = (\hat{\theta}(4\mathcal{I}(\mathbf{x}, \mathbf{y})), \mathbf{0}) = (4\hat{\theta}(\mathcal{I}(\mathbf{x}, \mathbf{y})), \mathbf{0})$$

and

$$[\theta(s, \mathbf{x}), \theta(t, \mathbf{y})] = [(*, \bar{\theta}(\mathbf{x})), (*, \bar{\theta}(\mathbf{y}))] = (4\mathcal{I}(\bar{\theta}(\mathbf{x}), \bar{\theta}(\mathbf{y})), \mathbf{0})$$

yield

$$\mathcal{I}(\bar{\theta}(\mathbf{x}), \bar{\theta}(\mathbf{y})) = \hat{\theta}(\mathcal{I}(\mathbf{x}, \mathbf{y})),$$

or, equivalently,

$$\left(\bar{\theta}(\mathbf{x})^t J_1 \bar{\theta}(\mathbf{y}), \bar{\theta}(\mathbf{x})^t J_2 \bar{\theta}(\mathbf{y}), \bar{\theta}(\mathbf{x})^t J_3 \bar{\theta}(\mathbf{y})\right)^t = \hat{\theta} \cdot \left(\mathbf{x}^t J_1 \mathbf{y}, \mathbf{x}^t J_2 \mathbf{y}, \mathbf{x}^t J_3 \mathbf{y}\right)^t$$

for all $\mathbf{x}, \mathbf{y}$. This happens if and only if $\bar{\theta}^t \mathbf{J} \bar{\theta} = \hat{\theta} \mathbf{J}$.

Conversely, suppose that $\bar{\theta} \in O(\mathbf{J}; 2, 2)$, i.e., $\bar{\theta}^t \mathbf{J} \bar{\theta} = \lambda \mathbf{J}$ is satisfied for some $\lambda \in \mathrm{GL}(3, \mathbb{R})$. We define $\theta \in \mathrm{Aut}(\mathbb{R}^3 \tilde{\times} \mathbb{R}^4)$ by

$$\theta(s, \mathbf{x}) = (\lambda \cdot s, \bar{\theta}(\mathbf{x})).$$

Then

$$\begin{aligned}
\theta((s,\mathbf{x})\cdot(t,\mathbf{y})) &= \theta(s+t+2\mathcal{I}(\mathbf{x},\mathbf{y}),\mathbf{x}+\mathbf{y})\\
&= (\lambda\cdot(s+t+2\mathcal{I}(\mathbf{x},\mathbf{y})),\bar\theta(\mathbf{x}+\mathbf{y}))\\
&= (\lambda\cdot s+\lambda\cdot t+\lambda\cdot 2\mathcal{I}(\mathbf{x},\mathbf{y}),\bar\theta(\mathbf{x}+\mathbf{y})),\\
\theta(s,\mathbf{x})\cdot\theta(t,\mathbf{y}) &= (\lambda\cdot s,\bar\theta(\mathbf{x}))\cdot(\lambda\cdot t,\bar\theta(\mathbf{y}))\\
&= (\lambda\cdot s+\lambda\cdot t+2\mathcal{I}(\bar\theta(\mathbf{x}),\bar\theta(\mathbf{y})),\bar\theta(\mathbf{x})+\bar\theta(\mathbf{y})).
\end{aligned}$$

Now the condition $\bar\theta^t\mathbf{J}\bar\theta=\lambda\mathbf{J}$ guarantees that

$$\lambda\cdot\mathcal{I}(\mathbf{x},\mathbf{y})=\mathcal{I}(\bar\theta(\mathbf{x}),\bar\theta(\mathbf{y})).$$

Thus θ is an automorphism of $\mathbb{R}^3\tilde{\times}\mathbb{R}^4$. Moreover, this defines a split homomorphism $O(\mathbf{J};2,2)\to\mathrm{Aut}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4)$. □

Proposition 2.2. (Structure of $\mathrm{Aut}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4)$)

$$\mathrm{Aut}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4)\cong\mathrm{Hom}(\mathbb{R}^4,\mathbb{R}^3)\rtimes O(\mathbf{J};2,2)$$

where an element $(\eta,A)\in\mathrm{Hom}(\mathbb{R}^4,\mathbb{R}^3)\rtimes O(\mathbf{J};2,2)$ *acts by*

$$(\eta,A)(s,\mathbf{x})=(\hat{A}s+\eta(\mathbf{x}),A\mathbf{x}).$$

Proof. Let $\theta\in\mathrm{Aut}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4)$. Then we have the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3\tilde{\times}\mathbb{R}^4 & \longrightarrow & \mathbb{R}^4 & \longrightarrow & 1\\
 & & \Big\downarrow{\scriptstyle\hat\theta} & & \Big\downarrow{\scriptstyle\theta} & & \Big\downarrow{\scriptstyle\bar\theta} & & \\
1 & \longrightarrow & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3\tilde{\times}\mathbb{R}^4 & \longrightarrow & \mathbb{R}^4 & \longrightarrow & 1
\end{array}$$

Thus $\theta(s,\mathbf{x})=(\hat\theta(s)+\eta(s,\mathbf{x}),\bar\theta(\mathbf{x}))$ for $(s,\mathbf{x})\in\mathbb{R}^3\tilde{\times}\mathbb{R}^4$, where $\eta:\mathbb{R}^3\tilde{\times}\mathbb{R}^4\to\mathbb{R}^3$. Since θ is a homomorphism, one can show that η is a homomorphism, i.e.,

$$\eta((s,\mathbf{x})(t,\mathbf{y}))=\eta(s,\mathbf{x})+\eta(t,\mathbf{y}).$$

In particular, $(\hat\theta(s),\mathbf{0})=\theta(s,\mathbf{0})=(\hat\theta(s)+\eta(s,\mathbf{0}),\mathbf{0})$ implies that $\eta(s,\mathbf{0})=0$ for all $s\in\mathbb{R}^3$, and thus $\eta(s,\mathbf{x})=\eta((s,\mathbf{0})(0,\mathbf{x}))=\eta(s,\mathbf{0})+\eta(0,\mathbf{x})=\eta(0,\mathbf{x})$. Hence $\eta\in\mathrm{Hom}(\mathbb{R}^4,\mathbb{R}^3)$.

Let's find out the kernel of the surjective homomorphism of Lemma 2.1:

$$\mathrm{Aut}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4)\to O(\mathbf{J};2,2),\quad \theta\mapsto\bar\theta.$$

Suppose that $\theta\in\mathrm{Aut}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4)$ with $\bar\theta=\mathrm{id}_{\mathbb{R}^4}$. Then $\hat\theta=\mathrm{id}_{\mathbb{R}^3}$ and thus $\theta(s,\mathbf{x})=(s+\eta(\mathbf{x}),\mathbf{x})$ for some $\eta\in\mathrm{Hom}(\mathbb{R}^4,\mathbb{R}^3)$. Conversely given $\eta\in\mathrm{Hom}(\mathbb{R}^4,\mathbb{R}^3)$, define $\theta\in\mathrm{Aut}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4)$ by $\theta(s,\mathbf{x})=(s+\eta(\mathbf{x}),\mathbf{x})$. Clearly this θ lies in the kernel of the homomorphism. Hence we have a short exact sequence

$$1\to\mathrm{Hom}(\mathbb{R}^4,\mathbb{R}^3)\to\mathrm{Aut}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4)\to O(\mathbf{J};2,2)\to 1.$$

By Lemma 2.1, this sequence is split. □

Note that $\mathrm{Hom}(\mathbb{R}^4,\mathbb{R}^3) \rtimes O(\mathbf{J};2,2)$ is sitting in

$$\mathrm{Hom}(\mathbb{R}^4,\mathbb{R}^3) \rtimes (\mathrm{GL}(3,\mathbb{R}) \times O(\mathbf{J};2,2))$$

as $(\eta,(\hat{A},A))$, and the action of $O(\mathbf{J};2,2)$ on $\mathrm{Hom}(\mathbb{R}^4,\mathbb{R}^3)$ is

$${}^{A}\eta(\mathbf{x}) = \hat{A}\cdot\eta(A^{-1}\mathbf{x}).$$

The group operation on $(\mathbb{R}^3\tilde{\times}\mathbb{R}^4) \rtimes O(\mathbf{J};2,2)$ is given by

$$\begin{aligned} ((s,\mathbf{x}),A)((t,\mathbf{y}),B) &= ((s,\mathbf{x})\cdot\ {}^{A}(t,\mathbf{y}),AB) \\ &= ((s,\mathbf{x})\cdot(\hat{A}t,A\mathbf{y}),AB) \\ &= ((s+\hat{A}t+2\mathcal{I}(\mathbf{x},A\mathbf{y}),\mathbf{x}+A\mathbf{y}),AB). \end{aligned}$$

3. The structure of AB-groups for $\mathcal{H}_7(\mathbb{H})$

Let $\mathit{\Pi} \subset \mathbb{R}^3\tilde{\times}\mathbb{R}^4 \rtimes \mathrm{Aut}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4)$ be an AB-group. Then it is well known that $\Gamma = \mathit{\Pi} \cap (\mathbb{R}^3\tilde{\times}\mathbb{R}^4)$, the pure translations in $\mathit{\Pi}$, is the maximal normal nilpotent subgroup, and $\Phi = \mathit{\Pi}/\Gamma$, the holonomy group of $\mathit{\Pi}$, is finite. Since Γ is a lattice of $\mathbb{R}^3\tilde{\times}\mathbb{R}^4$, $\Gamma\cap\mathcal{Z}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4)$ is a lattice of $\mathcal{Z}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4) = \mathbb{R}^3$, and $\Gamma/\Gamma\cap\mathcal{Z}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4)$ is a lattice of $\mathbb{R}^3\tilde{\times}\mathbb{R}^4/\mathcal{Z}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4) = \mathbb{R}^4$. Thus

$$\Gamma\cap\mathcal{Z}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4) \cong \mathbb{Z}^3$$

and

$$\Gamma/\Gamma\cap\mathcal{Z}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4) \cong \mathbb{Z}^4.$$

Consider the following natural commutative diagram (I):

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{R}^3 & \xrightarrow[=]{} & \mathbb{R}^3 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathbb{R}^3\tilde{\times}\mathbb{R}^4 & \longrightarrow & \mathbb{R}^3\tilde{\times}\mathbb{R}^4 \rtimes \mathrm{Aut}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4) & \longrightarrow & \mathrm{Aut}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{=} & & \\
1 & \longrightarrow & \mathbb{R}^4 & \longrightarrow & \mathbb{R}^4 \rtimes \mathrm{Aut}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4) & \longrightarrow & \mathrm{Aut}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

Recall from Proposition 2.2 that an element

$$(\eta,A) \in \mathrm{Aut}(\mathbb{R}^3\tilde{\times}\mathbb{R}^4) = \mathrm{Hom}(\mathbb{R}^4,\mathbb{R}^3) \rtimes O(\mathbf{J};2,2)$$

acts on $(s,\mathbf{x}) \in \mathbb{R}^3\tilde{\times}\mathbb{R}^4$ by

$$(\eta,A)(s,\mathbf{x}) = (\hat{A}s+\eta(\mathbf{x}),A\mathbf{x}).$$

Thus $O(\mathbf{J};2,2)$ acts on $\mathbb{R}^3$ via the homomorphism $\widehat{\ }: O(\mathbf{J};2,2) \to \mathrm{GL}(3,\mathbb{R})$, and $O(\mathbf{J};2,2)$ acts on $\mathbb{R}^4$ by matrix multiplication $O(\mathbf{J};2,2)\times\mathbb{R}^4 \to \mathbb{R}^4$.

Let $Q = \mathit{\Pi}/\mathbb{Z}^3$. Then the above diagram induces the following commutative diagram (II):

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{Z}^3 & \xrightarrow[=]{} & \mathbb{Z}^3 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Gamma & \longrightarrow & \mathit{\Pi} & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{=} & & \\
1 & \longrightarrow & \mathbb{Z}^4 & \longrightarrow & Q & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
$$

Here $\Phi \subset O(\mathbf{J}; 2, 2)$ acts on $\mathbb{Z}^4$ by matrix multiplication, and on $\mathbb{Z}^3$ via the homomorphism $\widehat{\ }\colon O(\mathbf{J}; 2, 2) \to \mathrm{GL}(3, \mathbb{R})$.

Lemma 3.1. *Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ be a basis of $\mathbb{R}^4$ of rational entries. Then for each $i = 1, 2, 3, 4$, the set $\{\mathcal{I}(\mathbf{v}_i, \mathbf{v}_j) \mid j = 1, \cdots, \widehat{i}, \cdots, 4\}$ forms a lattice of $\mathbb{R}^3$.*

Proof. Let V be the 4×4-matrix whose column vectors are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, and let W_i be the 3×3-matrix whose column vectors are $\mathcal{I}(\mathbf{v}_i, \mathbf{v}_j)$ with $j = 1, \cdots, \widehat{i}, \cdots, 4$. Then we can show that

$$\det(W_1) = \det[\mathcal{I}(\mathbf{v}_1, \mathbf{v}_2), \mathcal{I}(\mathbf{v}_1, \mathbf{v}_3), \mathcal{I}(\mathbf{v}_1, \mathbf{v}_4)] = -|\mathbf{v}_1|^2 \det[\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3 \mathbf{v}_4].$$

Similarly, $\det(W_i) = \pm|\mathbf{v}_i|^2 \det(V)$. Thus if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ is a basis of $\mathbb{R}^4$, then the set $\{\mathcal{I}(\mathbf{v}_i, \mathbf{v}_j) \mid j = 1, \cdots, \widehat{i}, \cdots, 4\}$ spans $\mathbb{R}^3$. The map $\mathcal{I}(\mathbf{x}, \mathbf{y})$ consists of polynomial functions of the entries of $\mathbf{x}$ and $\mathbf{y}$. Therefore the group generated by $\{\mathcal{I}(\mathbf{v}_i, \mathbf{v}_j) \mid j = 1, \cdots, \widehat{i}, \cdots, 4\}$ is discrete so that it forms a lattice of $\mathbb{R}^3$. ☐

This lemma tells us that the lattice $\mathbb{Z}^4$ of $\mathbb{R}^4$ generated by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ yields a lattice generated by $\{4\mathcal{I}(\mathbf{v}_i, \mathbf{v}_j) \mid j = 1, \cdots, \widehat{i}, \cdots, 4\}$, which must be contained in $\mathbb{Z}^3$ in the above diagram. However, the $\mathbb{Z}^3$ in the diagram can be finer than the lattice generated by $\{4\mathcal{I}(\mathbf{v}_i, \mathbf{v}_j) \mid j = 1, \cdots, \widehat{i}, \cdots, 4\}$.

It is interesting to see that

$$O(\mathbf{J}; 2, 2) = SO_0(2, 2) \times \mathbb{R}^+,$$

and the homomorphism

$$\widehat{\ }\colon O(\mathbf{J}; 2, 2) \to \mathrm{GL}(3, \mathbb{R})$$

has image $SO_0(1, 2) \times \mathbb{R}^+$. Note that if $C \in \mathrm{GL}(n, \mathbb{R})$ has finite order, then $\det(C) = \pm 1$ and every eigenvalue of C has modulus 1.

Recall from [5, Proposition 2] that a virtually free abelian group

$$1 \to \mathbb{Z}^4 \to Q \to \Phi \to 1$$

is a crystallographic group if and only if the centralizer of $\mathbb{Z}^4$ in Q has no torsion elements. Since Φ acts effectively on $\mathbb{Z}^4$, it follows that Q is naturally a 4-dimensional crystallographic group.

Let Π be an AB-group for $\mathcal{H}_7(\mathbb{H}) = \mathbb{R}^3 \tilde{\times} \mathbb{R}^4$. Then Π is a torsion free extension of $\mathbb{Z}^3$ by a 4-dimensional crystallographic group Q so that

$$1 \longrightarrow \mathbb{Z}^3 \longrightarrow \Pi \longrightarrow Q \longrightarrow 1$$

is exact.

Construction from Q For each Q (4-dimensional crystallographic group), we shall check if there exists a construction from Q; that is, a torsion free $\Pi \subset \mathcal{H}_7(\mathbb{H}) \rtimes \mathrm{Aut}(\mathcal{H}_7(\mathbb{H}))$ fitting the short exact sequence

$$1 \longrightarrow \mathbb{Z}^3 \longrightarrow \Pi \longrightarrow Q \longrightarrow 1.$$

This is the key notion for our arguments and construction. We have a complete classification of 4-dimensional crystallographic groups (Q's in the above statement). We shall use the presentations of the 4-dimensional crystallographic groups given in the book [1]:

> H. Brown, R. Bülow, J. Neubüser, H. Wondratschek and H. Zassenhaus, *Crystallographic Groups of Four-Dimensional Spaces*, John Wiley & Sons, New York, 1978.

The crystallographic groups will be called Q, and every Q has an explicit representation $Q \longrightarrow \mathbb{R}^4 \rtimes \mathrm{GL}(4, \mathbb{Z})$ in this book.

Our goal is to determine which Q will give rise to a *torsion free* Π that fits the commutative diagram (II). When Q is torsion free, then Π will be automatically torsion free, but when Q contains a torsion subgroup Q_0, we need to check whether the lift Q_0 to Π will be torsion free.

Let Π_0 be the lift of Q_0 to Π. Thus

$$1 \longrightarrow Z \longrightarrow \Pi_0 \longrightarrow Q_0 \longrightarrow 1$$

is exact. If Π is torsion free, then so is Π_0, and is a 3-dimensional Bieberbach group. Therefore we have the following theorem [7] and we shall classify all infra-nilmanifolds with $\mathcal{H}_7(\mathbb{H})$-geometry.

Theorem 3.2. *For a 4-dimensional crystallographic group Q to have a construction, its holonomy group Ψ must be in $SO_0(2,2)$. Therefore, if Ψ contains a matrix of determinant -1, there is no construction from Q.*

Recall that the 6-dimensional nilpotent Lie group $\mathfrak{N}^3 = \mathfrak{so}(3) \times \mathbb{R}^3$ has group law

$$(s, \mathbf{x})(t, \mathbf{y}) = (s + t + \mathcal{I}(\mathbf{x}, \mathbf{y}), \mathbf{x} + \mathbf{y}),$$

where

$$\mathcal{I}(\mathbf{x},\mathbf{y}) = \mathbf{x}\mathbf{y}^t - \mathbf{y}\mathbf{x}^t = \begin{bmatrix} 0 & x_1y_2 - x_2y_1 & x_1y_3 - x_3y_1 \\ x_2y_1 - x_1y_2 & 0 & x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 & x_3y_2 - x_2y_3 & 0 \end{bmatrix}.$$

If we identify $\mathfrak{so}(3)$ with $\mathbb{R}^3$ by

$$\begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix} \longleftrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

then clearly,

$$\mathcal{I}(\mathbf{x},\mathbf{y}) = \mathbf{x} \times \mathbf{y} \text{ (cross product)}$$

so that $\mathfrak{R}^3 = \mathfrak{so}(3) \times \mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^3$ has the group operation

$$(s,\mathbf{x})(t,\mathbf{y}) = (s + t + \mathbf{x} \times \mathbf{y},\ \mathbf{x} + \mathbf{y}).$$

If we follow a similar procedure in $\mathfrak{R}^3 = \mathfrak{so}(3) \times \mathbb{R}^3$ such as $\mathcal{H}_7(\mathbb{H})$, then we have the following theorem [6].

Theorem 3.3. *There exists an almost Bieberbach group $\Pi \subset \mathfrak{R}^3 \rtimes \mathrm{Aut}(\mathfrak{R}^3)$ whose holonomy group is $D_8 \times \mathbb{Z}_2$, of order 16.*

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