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RECENT PROGRESS IN SEMIALGEBRAIC TRANSFORMATION GROUP THEORY WITH SOME APPLICATIONS

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ABSTRACT. We survey some recent developments in semialgebraic transformation group theory which has been turned out to be useful for the study of difficult classical problems.

CONTENTS

INTRODUCTION

In "On Semialgebraic Transformation Groups I, II" [1] [2], the authors discussed very shortly the theory of semialgebraic transformation groups. In this article, which might be a continuation of [1] [2], we introduce some recent results especially focused on the applications.

Recall the class of *semialgebraic sets* in $\mathbb{R}^n$ is the smallest collection of subsets containing all $\{x \in \mathbb{R}^n \mid p(x) > 0\}$ for each real valued polynomial $p(x) = p(x_1, \dots, x_n)$ which is stable under finite union, finite intersection and complement. A map $f\colon M \to N$ between semialgebraic sets $M(\subset \mathbb{R}^m)$ and $N(\subset \mathbb{R}^n)$ is called a *semialgebraic map* if its graph is a semialgebraic set in $\mathbb{R}^m \times \mathbb{R}^n$. From now on we impose "euclidean topology" on semialgebraic sets and mainly consider continuous semialgebraic maps.

From its definition, semialgebraic sets are located between real algebraic sets and topological spaces. So the theory of semialgebraic sets are applicable to the study of real algebraic sets and to that of topological spaces. Most founders of semialgebraic geometry focused on the first direction of the applications: Especially, J. Bochnak, M. Coste, and M.-F. Roy obtained many useful results about real algebraic sets by the use of semialgebraic theory [3].

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On the other hand, S. Łojasiewicz [9] and H. Hironaka [8] proved the semialgebraic triangulation theorem which implies that in particular if a semialgebraic set is compact then it has a triangulation. So the application of the theory of semialgebraic sets to that of the topological spaces might have little advantage since we have already well established theory of "PL spaces". However, when we consider the equivariant theory, the story changes drastically.

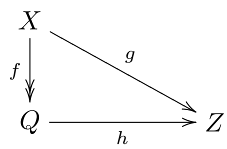
The authors of this article are mainly interested in transformation group theory. As for the nonequivariant case, the equivariant semialgebraic sets lie between real equivariant algebraic sets and equivariant topological spaces. Contrast to the nonequivariant case, we focus on the application of equivariant semialgebraic spaces to the study of topological space since we have noticed that semialgebraic geometry provides a combinatorial method for the study of equivariant topological spaces.

In Section 1 we shall review some background materials of semialgebraic sets especially focused on the nice picture of them. In Section 2 we survey some results about semialgebraic transformation theory. These two sections are continuation of "On Semialgebraic Transformation Groups I, II" [1] [2]. Finally, we survey some applications of the semialgebraic geometry to the problem of topological transformation theory.

1. SEMIALGEBRAIC SETS

In this section we gather some plausible aspects of semialgebraic sets. For more valuable results, we refer the reader to [1] or [3].

Proposition 1.1. (1) *Every semialgebraic set has a finite number of connected components which are semialgebraic.*
(2) *The composition of two semialgebraic maps is semialgebraic.*
(3) *(Tarski-Seidenberg principle) Let $f\colon X \to Y$ be a semialgebraic map. If $A \subset X$ is semialgebraic, then its image $f(A)$ is semialgebraic. If $B \subset Y$ is semialgebraic, then its inverse image $f^{-1}(B)$ is semialgebraic.*
(4) *If $A \subset X$ is semialgebraic, then the closure $\overline{A}$ and the interior $\mathring{A}$ in X are all semialgebraic.*
(5) *Let $f\colon X \to Q$ and $g\colon X \to Y$ be semialgebraic. Assume f is surjective. If $h\colon Q \to Y$ is a map such that $h \circ f = g$, then h is semialgebraic.*



The Tarski-Seidenberg principle of the above proposition is one of advantages of the category of semialgebraic sets. Another important picture of semialgebraic sets is the semialgebraic triangulation theorem.

A *finite open simplicial complex* $(K, (\sigma_i \mid i \in I))$ is defined as a subset of some $\mathbb{R}^m$ equipped with a partition $(\sigma_i \mid i \in I)$ composed of a finite number of open simplices σ_i in $\mathbb{R}^m$, such that the intersection $\bar{\sigma}_i \cap \bar{\sigma}_j$ of the closures of any two open simplices σ_i and σ_j is either empty or a common face of $\bar{\sigma}_i$ and $\bar{\sigma}_j$. Thus a finite open simplicial complex

$(K, (\sigma_i))$ is obtained from some finite, hence compact, simplicial complex L by deleting some open simplices σ of L. S. Łojasiewicz [9] and H. Horonaka [8] proved the following semialgebraic triangulation theorem.

Theorem 1.2 (Semialgebraic triangulation)**.** *Let M be a semialgebraic set and let $M_1, \ldots, M_k$ be semialgebraic subsets of M. Then there exist a finite open simplicial complex K and a semialgebraic homeomorphism $\tau\colon |K| \to M$ such that each M_i is a finite union of some of the $\tau(\sigma)$, where σ is an open simplex of K.*

In the above theorem, if M is compact, then the open simplicial complex K will be a finite simplicial complex in the classical sense.

Concerning the triangulation of maps, it is known that semialgebraic function on a compact semialgebraic set is semialgebraically triangulable, more precisely, if $f\colon X \to \mathbb{R}$ is a continuous semialgebraic function from a compact semialgebraic set X to $\mathbb{R}$, then there exist a semialgebraic triangulations of X and $\mathbb{R}$ so that f is a simplicial map (see [3, Theorem 9.4.1].) For general case we cannot expect the similar result even if X is compact and the target of f is $\mathbb{R}^2$. Consider, for instance, the map $f\colon [0,1]^2 \to \mathbb{R}^2$ defined by $f(x,y) = (x, xy)$. If there were a semialgebraic triangulation of f, a regular neighborhood of $\{0\} \times I$ would be mapped into a semialgebraic set of dimension less than or equal to 1, which leads to a contradiction.

However, there is a weaker theorem which describes a nice feature of semialgebraic maps.

Theorem 1.3 (Semialgebraic triviality)**.** *Let X and Y be two semialgebraic sets, $f\colon X \to Y$ a continuous semialgebraic map. Then there exit a finite partition of semialgebraic subsets of Y into semialgebraic sets $Y = \bigcup_{l=1}^{r} Y_l$ and, for each l, a semialgebraic trivialization $\theta_l\colon Y_l \times F_l \to f^{-1}(Y_l)$ of f over Y_l. Here, the semialgebraic trivialization means that θ_l is a semialgebraic homeomorphism preserving the fibers, and F_l denotes a fiber over Y_l.*

Remark 1.4. We can consider *semialgebraic spaces* as objects obtained by pasting finitely many semialgebraic sets together along open semialgebraic subsets. R. Robson [14] proves that every 'regular' semialgebraic space admits a semialgebraic embedding into $\mathbb{R}^n$ for some n. This explains why we only consider semialgebraic sets instead of semialgebraic spaces.

Before we close this section, we introduce "Nash" manifolds.

A *Nash function* is a real analytic function $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$ (where U is an open semialgebraic subset of $\mathbb{R}^n$) which is algebraic, that is, there is a nontrivial polynomial P such that $P(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) = 0$ for all $(x_1, \ldots, x_n) \in U$. Equivalently, a Nash function is a function that is at once analytic and semialgebraic. An n-dimensional manifold M is called *Nash* if M is covered by a finite number of patches (U_i, ψ_j) such that $\psi_j \circ \psi_i^{-1}$ and $\psi_i \circ \psi_j^{-1}$ are Nash where they are defined. Equivalently, a Nash manifold is a smooth semialgebraic space.

2. Semialgebraic groups and Semialgebraic G-sets

The definition of a semialgebraic group is similar to that of a Lie group, i.e., a semialgebraic set $G \subset \mathbb{R}^n$ is called a *semialgebraic group* if it is a topological group such that the group multiplication and the inversion are continuous and semialgebraic. Note that every compact semialgebraic group has a Lie group structure. Conversely, every compact Lie group has a semialgebraic group structure.

By a *semialgebraic transformation group* we mean a triple (G, M, θ), where G is a semialgebraic group, M is a semialgebraic set, and $\theta\colon G \times M \to M$ is a continuous semialgebraic map such that:

(1) $\theta(g, \theta(h, x)) = \theta(gh, x)$ for all $g, h \in G$ and $x \in M$
(2) $\theta(e, x) = x$ for all $x \in M$, where e is the identity of G.

In this case M is called a *semialgebraic G-set*. As usual we shortly write gx for $\theta(g, x)$. A *semialgebraic G-subset* of a semialgebraic G-set M is a G-invariant semialgebraic subset of M. A continuous semialgebraic map $f\colon M \to N$ between semialgebraic G-sets is said to be a *semialgebraic G-map* if it is equivariant, i.e., $f(gx) = gf(x)$ for all $g \in G$ and $x \in M$.

Remark 2.1. In our definition of semialgebraic groups, we assumed multiplications and inversions are continuous. In general, we can define a semialgebraic group without assuming the continuity. A. Pillary has shown [13, Remark 2. 6] that if G is a semialgebraic group in the sense that G has discontinuous semialgebraic multiplication and inversion then it is possible to find a finite collection of semialgebraic subsets $U_i \subset G$ and semialgebraic charts $\psi_i\colon U_i \to \mathbb{R}^n$ that induce on G the structure of a Nash group. Moreover, such Nash structure is unique. Thus there is no harm to define a semialgebraic group to have continuous multiplication and inversion.

On the other hand, there are many Nash groups [10] which can not be embedded in $\mathbb{R}^n$ while every semialgebraic group can be imbedded in $\mathbb{R}^n$ by Remark 1.4.

Now we shortly sketch the picture of semialgebraic orbit spaces and orbit types.

Theorem 2.2 ([5])**.** *Let G be a compact semialgebraic group and M a semialgebraic G-set, then M/G is a semialgebraic set and the orbit map is semialgebraic.*

Let G be a compact semialgebraic group. For a semialgebraic G-set M and a point x of M we can associate an orbit type (G/G_x) which is denoted by $\mathrm{type}(G/G_x)$. We say that $x, y \in M$ have the same orbit type if G_y is conjugate to G_x. We call the association $x \mapsto \mathrm{type}(G/G_x)$ the *orbit structure* of M.

Proposition 2.3 ([12, Theorem 3.2], [6, Theorem 2.6])**.** *Every semialgebraic G-set has finitely many orbit types.*

The association $x \in M$ to its orbit type $\mathrm{type}(G/G_x)$ naturally induces to an association $\bar{x} \in M/G$ to $\mathrm{type}(G/G_x)$, we understand it as an orbit structure of the orbit space M/G unless specified otherwise.

The existence of slices plays an important role in the development of transformation group theory. There is a similar result in the semialgebraic equivariant theory as follows.

Theorem 2.4 ([6, Theorem 1.1])**.** *Let G be a compact semialgebraic group and M a semialgebraic G-set. Then*

(1) *for each $x \in M$, there is a semialgebraic G_x-slice S at x, and*
(2) *M can be covered by a finite number of semialgebraic G-tubes.*

3. Covering homotopy theorem and Mapping cylinder conjecture

The covering homotopy theorem is a key theorem for classifying G-spaces over a fixed orbit space X much similar to the classifying bundles by the use of homotopies. The original covering homotopy theorem is stated as follows:

Theorem 3.1 ([4, Theorem II.7.3] or [11])**.** *Let G be a compact Lie group and let X and Y be G-spaces. Assume that every open subspace of X/G is paracompact. Let $f\colon X \to Y$ be equivariant and let $f'\colon X/G \to Y/G$ be the induced map. Let $F'\colon X/G \times I \to Y/G$ be a homotopy which preserves the orbit structure and starts at f' (thus, f' preserves the orbit types). Then there exists an equivariant homotopy $F\colon X \times I \to Y$ starting at f and covering F', i.e., $\pi_Y \circ F = F' \circ \pi_{X\times I}$.*

If both X and Y in the above theorem have only one orbit type $\mathrm{type}(G/H)$, then X and Y can be seen as bundles with the fiber G/H and the structure group $N(H)/H$. The property "preserving orbit structure" will become the property that the corresponding maps are bundle maps.

Now we want to translate the covering homotopy theorem into that in the semialgebraic category.

Theorem 3.2. *Let G be a compact semialgebraic group and let X and Y be semialgebraic G-sets. Let $f\colon X \to Y$ be a continuous semialgebraic G-map and let $f'\colon X/G \to Y/G$ be the induced map. Let $F'\colon X/G\times I \to Y/G$ be a semialgebraic homotopy which preserves the orbit structure. Then there exists an semialgebraic G-homotopy $F\colon X \times I \to Y$ covering F' and starting at f'. (Again, of course, f, f' and F will preserve the orbit structures.)*

The above theorem is proved in [7] with its relative form.

The condition of "preserving orbit types" might be too strong to handle enough cases of G-spaces. For example, an isolated fixed point of a smooth G-manifold has a G-retractible neighborhood but the contraction is not orbit type preserving. So Bredon tried to weaken the condition, and he conjectured the following:

Conjecture 3.3 ([4, p. 98])**.** Let G be a compact Lie group, and W a compact G-space. Suppose that W/G has the form of a mapping cylinder with orbit types constant along generators of the cylinder less than the base. Then we conjecture that W is equivalent to a mapping cylinder of an equivariant map inducing the given mapping cylinder structure on W/G.

As the simplest case we can consider a G-space W whose orbit structure is conical, i.e., the orbit space W/G is a conic in the conjecture. Even for this case, only partial solutions are known, also see [4, Section II.8].

To translate the conjecture to the problem in the semialgebraic category, we need a new notion. A semialgebraic map $f\colon X \to Y$ is called *semialgebraically proper* if $f^{-1}(C)$ is compact for every compact semialgebraic subset C of Y. Since C should be semialgebraic in the definition, this notion is weaker than the property that f is semialgebraic *and* proper. In the semialgebraic category, if the map is semialgebraically proper, there is no difficulty to define mapping cylinder, so we assume that a mapping cylinder is defined by a semialgebraically proper map.

Theorem 3.4 ([7])**.** *Let W be a semialgebraic G-set. Suppose that W/G has the form of a semialgebraic mapping cylinder with orbit types constant along generators of the cylinder less the base. Then W is semialgebraically equivalent to a semialgebraic mapping cylinder of a semialgebraically proper equivariant map inducing the given semialgebraic mapping cylinder structure on W/G.*

Remark 3.5. If W is compact in the above theorem, then every related semialgebraic map will be naturally semialgebraically proper. Thus Theorem 3.4 gives an extended solution of Conjecture 3.3 about semialgebraic G-sets.

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