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STANDARDIZING CURVES IN PUNCTURED DISKS

SANG JIN LEE

Abstract. Let D^2 be the disc in the complex plane centered at the origin with radius $n+1$. Let D_n be the n-punctured disc $D^2 \setminus \{1, \dots, n\}$. We show that given any collection of essential simple closed curves in D_n there is a unique shortest positive n-braid that make it standard, that is each curve intersects the real line exactly twice.

1. Introduction

Let D^2 be the disc $\{z \in \mathbb{C} : |z| \le n+1\}$ in the complex plane and D_n be the n-punctured disc $D^2 \setminus \{1, \dots, n\}$. A *curve system* in D_n means a collection of simple closed curves in D_n. It is *essential* if none of its components is homotopic to a point, to a puncture or to the boundary. It is *standard* if each component intersects the real line exactly twice as Figure 1. We say that an an automorphism (orientation preserving self-homeomorphism) f of D_n *standardize* the curve system $\mathcal{C} \subset D_n$, if $f(\mathcal{C})$ is standard. See Figure 2. It is obvious that for any curve system in D_n there are infinitely many standardizing automorphisms. We are interested at:

Question. *Is there a canonical way to standardize essential curve systems in D_n?*

Figure 1. A standard curve system in D_{10}

The Artin braid group on n-strands, B_n, is the group of automorphisms of D_n that fix the boundary point-wise, modulo isotopy relative to the boundary. To answer the question, we study the action of braids on the curve systems in

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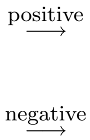
FIGURE 2. Standardization of a curve system



(a) Dehn-tiwsts

(b) Artin generator $\sigma_2 \in B_4$

FIGURE 3. Dehn-twists and Artin generator

punctured disks. Let σ_i be the isotopy class of the positive Dehn-twist along the straight line segment connecting the punctures i and $i+1$. See Figure 3 (a) for Dehn-twists. Note that any automorphism of D_n fixing the boundary point-wise is isotopic to a composition of $\sigma_i^{\pm 1}$. So σ_i's generate B_n. In fact B_n has the group presentation

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{cl} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } |i-j| \geq 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{if } |i-j| = 1 \end{array} \right\rangle .$$

An n-braid can be thought as a collection of n strands $l = l_1 \cup \cdots \cup l_n$ in the horizontal cylinder $[0,1] \times D^2$ such that $|\, l \cap (t \times D^2)| = n$ for $0 \leq t \leq 1$ and $l \cap (t \times D^2) = t \times \{1, \ldots, n\}$ for $t = 0, 1$, by considering the trajectories of the punctures under the isotopy from the the homeomorphism to the identity. Figure 3 (b) illustrates the correspondence.

A braid is *reducible* if there is a collection of essential simple closed curves whose isotopy class is invariant under the action of the given braid. The isotopy class of such a curve system is called a *reduction system*. Our question is related to the problems:

(1) Given a braid, decide whether or not it is reducible.
(2) Given a reducible braid, find a reduction system.

The importance and the interest of these questions come from the Nielsen-Thurston classification theorem [Thu88]. For a reducible braid, we can cut the disc along a reduction system to get a collection of simpler braids. We can do the same thing to the automorphisms of any surfaces. The Nielsen-Thurston classification theorem is that the irreducible automorphisms are either periodic or pseudo-Anosov. In pseudo-Anosov case, there is a beautiful dynamical structure, a pair of transverse invariant measured foliations. In reducible case, there is a canonical reduction system due to Birman, Lubotzky and McCarthy [BLM83] and Ivanov [Iva92].

There have been several approaches to decide the dynamical types of mapping classes. In [BH95], Bestvina and Handel make so-called the *train track algorithm* that decides, given any automorphism of orientable surfaces, the dynamical types and finds the dynamical structures; a pair of transverse invariant measured foliations for pseudo-Anosov case and a reduction system for reducible case. Independently in [BNG95], Bernardete, Nitecki and Gutiérrez make a different algorithm to decide the reducibility of braids.

All the above algorithms seem to be exponential with respect to the word-length of the given braid. For the train track algorithm, we first write the braid to a graph map, which is exponential. And the solution of Bernardete, Nitecki and Gutiérrez needs the whole algorithm to the conjugacy problem, which is also exponential up to the current knowledge. We hope that this paper provide a good step toward polynomial time (with respect to the braid index and the word-length of the given braid) solutions to decide the dynamical types and to find dynamical structures of braids.

The difficulty of recognizing reducibility comes from that the reduction systems are in general very complicated. But it is easy to decide whether a given braid has a standard reduction system.

Theorem 6. *There is an algorithm that decides, given an n-braid α, whether there exists a standard curve system that is invariant under α, and if there exists, finds one. The time complexity is $\mathcal{O}(n^3\ell^2)$, where ℓ is the word-length of α.*

Note that any reducible braid is conjugate to a braid with a standard reduction system. The question is how to find such a conjugate braids. So we study how to standardize the curve systems. For a braid α and a curve system $\mathcal{C}$, let $\alpha * \mathcal{C}$ be the result of the action of α on $\mathcal{C}$, which is the question mentioned at the beginning. The following result answers the question.

Corollary 8. *Given any curve system $\mathcal{C}$, there exist a unique positive braid P such that $P * \mathcal{C}$ is standard and if Q is another positive braid with $Q * \mathcal{C}$ standard, then $Q = RP$ for some positive braid R.*

The 'uniqueness' of the theorem is very useful in applications. To explain the idea of our approach, we recall the Garside algorithm. It is first used by Garside in [Gar69] to solve the word and the conjugacy problem in braid groups. In this approach, we study the properties of the positive braid monoid B_n^+ and get a unique *normal form* as $\alpha = \Delta^u A_1 \cdots A_k$ for $\alpha \in B_n$, where Δ is the *Garside element*, $u \in \mathbb{Z}$ and A_i's are the *permutation braids*. We note that, informally, the normal form can be considered as a *horizontal decomposition* under the convention of drawing braids horizontally, whereas the decomposition using the reduction system is *vertical*. So these two decompositions fit together well.

After finishing the paper, the author found that some lemmas in this paper are already known to Bernardete, Nitecki and Gutiérrez. But we include the proofs of such lemmas, because it is written in different ways in their paper and our proof is much simpler than their due to the recent progress on the Garside structures of braid groups.

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2. Garside algorithm to conjugacy problem

This section reviews the Garside algorithm briefly. See [Gar69, DP99, ECHLPT] for details. Let B_n^+ be the monoid generated by $\sigma_1, \dots, \sigma_{n-1}$ with defining relations same as in §1. Then B_n^+ is a (left and right) cancellative monoid that embeds in the group B_n under the canonical homomorphism. The elements of B_n^+ are called *positive braids*. We use the capital letters $P,\ Q,\ R, \dots$ to denote positive braids.

Define the partial ordering $\preccurlyeq$ and $\succcurlyeq$ on the positive braids as follows: if $P = QR$ for positive braids P, Q, R, then we write $Q \preccurlyeq P$ and $P \succcurlyeq R$. We say that Q is a *left divisor* of P and R is a *right divisor* of P.

For any positive braids P and Q, there is unique *left lcm* $P \vee_L Q \in B_n^+$ such that $(P \vee_L Q) \succcurlyeq P$, $(P \vee_L Q) \succcurlyeq Q$ and for any positive braid R, $R \succcurlyeq P$ and $R \succcurlyeq Q$ imply $R \succcurlyeq (P \vee_L Q)$. And there is unique *right gcd* $P \wedge_R Q \in B_n^+$ such that $P \succcurlyeq (P \wedge_R Q)$, $Q \succcurlyeq (P \wedge_R Q)$ and for any positive braid R, $P \succcurlyeq R$ and $Q \succcurlyeq R$ imply $(P \wedge_R Q) \succcurlyeq R$. The *right lcm* $P \vee_R Q$ and *left gcd* $P \wedge_L Q$ are defined similarly.

The braid $\Delta = (\sigma_1 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1\sigma_2)\sigma_1$ as in Figure 4 (a) is called the *Garside element*. The set of left divisors of Δ equals the set of right divisors. The left (equivalently, right) divisors of Δ are called the *simple elements*. For

(a) Garside element $\Delta \in B_4$ (b) A braid in normal form

FIGURE 4. Garside element and normal form

each i, σ_i is a simple element. And $\Delta^{-1}\alpha\Delta = \tau(\alpha)$ for $\alpha \in B_n$, where τ is the automorphism of B_n such that $\tau(\sigma_i) = \sigma_{n-i}$.

For any $\alpha \in B_n$, there are integers $r \le s$ such that $\Delta^r \preccurlyeq \alpha \preccurlyeq \Delta^s$. The maximal such r is called the *infimum* and denoted by $\inf(\alpha)$. The minimal such s is called the *supremum* and dented by $\sup(\alpha)$. There are uniquely defined simple elements $A_1, \ldots, A_k$ such that $\alpha = \Delta^{\inf(\alpha)} A_1 \cdots A_k$ and $(A_i \cdots A_k) \wedge_L \Delta = A_i$. We call this expression as the *left normal form* of α. The *right normal form* is defined similarly: $\alpha = \Delta^{\inf(\alpha)} A_1 \cdots A_k$ and $(A_1 \cdots A_i) \wedge_R \Delta = A_i$. We sometimes write the left/right normal form as $\alpha = A_1 \cdots A_k \Delta^{\inf(\alpha)}$.

The normal form can be also characterized by using the *staring set* and the *finishing set*. For a positive braid P they are denoted by $S(P)$ and $F(P)$ respectively:

$$\begin{aligned} S(P) &= \{i \mid P = \sigma_i Q \quad \text{for some } Q \in B_n^+\} \\ F(P) &= \{i \mid P = Q\sigma_i \quad \text{for some } Q \in B_n^+\}. \end{aligned}$$

The simple elements are also called the *permutation braids*, because they are in one-to-one correspondence with the n-permutations: for a permutation θ on $\{1, \ldots, n\}$, connect $(0, \theta(i))$ to $(1, i)$ by a straight line and then at each crossing, make the line from $(0, \theta(i))$ to $(1, i)$ lie over the line from $(0, \theta(j))$ to $(1, j)$ if $i < j$. We remark some results concerning normal forms.

(1) A positive braid is a permutation braid if and only if any two strands cross at most once [ECHLPT, EM94].
(2) For a permutation braid A corresponding to a permutation θ, the starting set and finishing set are as follows:

$$\begin{aligned} S(A) &= \{i \mid \theta^{-1}(i) > \theta^{-1}(i+1)\} \\ F(A) &= \{i \mid \theta(i) > \theta(i+1)\}. \end{aligned}$$

(3) $\alpha = \Delta^r A_1 \cdots A_k$ is the left normal form if and only if $r = \inf(\alpha)$ and $F(A_i) \supset S(A_{i+1})$ for $i = 1, \ldots, k-1$. See Figure 4 (b).
(4) In [ECHLPT], Thurston introduced the *np*-form. Any braid $\alpha \in B_n$ has a unique expression, called the *np*-form, as $\alpha = P^{-1}Q$, where P, Q are positive braids with $S(P) \cap S(Q) = \emptyset$. In this case if $P \neq e$ and $\alpha = \Delta^r A_1 \cdots A_k$ is the left normal form, then $Q = A_i \cdots A_k$ for some i.

FIGURE 5. A un-nested standard curve system $\mathcal{C}_{\mathbf{n}}$, $\mathbf{n} = (1, 1, 2, 1, 2, 3)$

(a) $\sigma_1^{-1}\sigma_2 \in B_3$ (b) $\langle\sigma_1^{-1}\sigma_2\rangle_{\mathbf{n}} \in B_6$ for $\mathbf{n} = (2, 3, 1)$ (c) $e \oplus \sigma_1 \oplus \sigma_2$

FIGURE 6

Similarly, we can consider pn-form as $\alpha = PQ^{-1}$, where P, Q are positive braids with $F(P) \cap F(Q) = \emptyset$. In this case if $P \neq e$ and $\alpha = A_1 \cdots A_k \Delta^r$ is the right normal form, then $Q^{-1} = A_i \cdots A_k \Delta^r$ for some i.

3. STANDARDIZING CURVES IN PUNCTURED DISKS

An essential curve system is called *un-nested* if none of its component is contained in another component. See Figure 5. The un-nested standard curve systems in D_n are in one to one correspondence with the k-compositions of n for $k = 2, \ldots, n-1$. Recall that a tuple $\mathbf{n} = (n_1, \ldots, n_k)$ of positive integers is a *k-composition* of n if $n = n_1 + \cdots + n_k$. For a composition $\mathbf{n} = (n_1, \ldots, n_k)$, Let $\mathcal{C}_{\mathbf{n}}$ be the un-nested standard curve system $\mathcal{C} = \cup_{n_i \geq 2} C_i$, where C_i is the standard curve intersecting the real line once at each of the open intervals $(\sum_{j=1}^{i-1} n_j, 1+\sum_{j=1}^{i-1} n_j)$ and $(\sum_{j=1}^{i} n_j, 1+\sum_{j=1}^{i} n_j)$. See Figure 5.

For $\alpha_0 \in B_k$, we denote by $\langle\alpha_0\rangle_{\mathbf{n}}$ the n-braid obtained from α_0 by taking n_i parallel copies of the i-th strand of α_0. See Figure 6 (a,b). For a tuple $(\alpha_1, \ldots, \alpha_k)$, $\alpha_i \in B_{n_i}$, we denote by $\alpha_1 \oplus \cdots \oplus \alpha_k$ the n-braid $\alpha_1'\alpha_2' \cdots \alpha_k'$, where α_i' is the image of α_i under the homomorphism $B_{n_i} \to B_n$, $\sigma_j \mapsto \sigma_{n_1+\cdots+n_{i-1}+j}$. See Figure 6 (c), where e is the trivial element in B_1 which is the trivial group.

The n-braids act on the set of curve systems in D_n both on left and on right. We write $\mathcal{C} * \alpha$ and $\alpha * \mathcal{C}$ to denote the actions of an n-braid α to a curve system $\mathcal{C}$ in D_n. The k-braid group B_k acts on the set of k-compositions of n from left and right via the induced permutations as follows: for a k-composition $\mathbf{n} = (n_1, \cdots, n_k)$ and a k-braid α_0 with induced permutation θ, $\alpha_0 * \mathbf{n} = (n_{\theta^{-1}(1)}, \ldots, n_{\theta^{-1}(k)})$ and $\mathbf{n} * \alpha_0 = (n_{\theta(1)}, \ldots, n_{\theta(k)})$. It is easy to see the followings:

(1) If A is a permutation then so is $\langle A\rangle_{\mathbf{n}}$, since any two strands cross at most once.
(2) $\langle\alpha_0\rangle_{\mathbf{n}} * \mathcal{C}_{\mathbf{n}} = \mathcal{C}_{\alpha_0 * \mathbf{n}}$ and $C_{\mathbf{n}} * \langle\alpha_0\rangle_{\mathbf{n}*\alpha_0} = \mathcal{C}_{\mathbf{n}*\alpha_0}$.
(3) If A_i is the rightmost simple element of P_i for $i = 1, \ldots, k$, then $A_1 \oplus \cdots \oplus A_k$ is the rightmost simple element of $P_1 \oplus \cdots \oplus P_k$.
(4) $\langle\alpha_0\rangle_{\mathbf{n}}(\alpha_1 \oplus \cdots \oplus \alpha_k) = (\alpha_{\theta^{-1}(1)} \oplus \cdots \oplus \alpha_{\theta^{-1}(k)})\langle\alpha_0\rangle_{\mathbf{n}}$, where θ is the induced permutation α_0.

Lemma 1. *Let $\mathbf{n} = (n_1, \ldots, n_k)$ be a k-partition of n. If α is an n-braid such that $\alpha * \mathcal{C}_{\mathbf{n}}$ is un-nested and standard, then $\alpha = \langle\alpha_0\rangle_{\mathbf{n}}(\alpha_1 \oplus \cdots \oplus \alpha_k)$, for some k-braid α_0 and n_i-braids α_i for $i = 1, \ldots, k$. In particular, $\alpha * \mathcal{C}_{\mathbf{n}} = \mathcal{C}_{\alpha_0 * \mathbf{n}}$.*

Proof. Let E and E' be the outermost component of $D_n \setminus \mathcal{C}_{\mathbf{n}}$ and $D_n \setminus (\alpha * \mathcal{C}_{\mathbf{n}})$ respectively. Then E and E' are canonically identified with the k-punctured disc D_k since $\mathcal{C}_{\mathbf{n}}$ and $\alpha * \mathcal{C}_{\mathbf{n}}$ are un-nested and standard. And $\alpha(E) = E'$. Let α_0 be the k-braid obtained by the composition

$$D_k \simeq E \overset{\alpha|_E}{\to} E' \simeq D_k,$$

where $\alpha|_E$ is the restriction of α to E and $D_k \simeq E$ and $E' \simeq D_k$ are the identifications. Then α and $\langle\alpha_0\rangle_{\mathbf{n}}$ coincide on E so that $\langle\alpha_0\rangle_{\mathbf{n}}^{-1}\alpha$ is the identity on E. So $\langle\alpha_0\rangle_{\mathbf{n}}^{-1}\alpha = \alpha_1 \oplus \cdots \oplus \alpha_k$ for n_i-braids α_i. ☐

Lemma 2. *Let $\mathbf{n}$ be a k-composition of n and A_1, A_2 be permutation k-braids. If A_1A_2 is in left/right normal form, then so is $\langle A_1\rangle_{A_2*\mathbf{n}}\,\langle A_2\rangle_{\mathbf{n}}$.*

Proof. Let $A_2 * \mathbf{n} = (n'_1, \ldots, n'_k)$. Then it is easy to see that $F(\langle A_1\rangle_{A_2*\mathbf{n}}) = \{n'_1 + \cdots + n'_i \mid i \in F(A_1)\}$ and $S(\langle A_2\rangle_{\mathbf{n}}) = \{n'_1 + \cdots + n'_j \mid j \in S(A_2)\}$. Since A_1A_2 is in right normal form, $F(A_1) \subset S(A_2)$. So $F(\langle A_1\rangle_{A_2*\mathbf{n}}) \subset S(\langle A_2\rangle_{\mathbf{n}})$ and $\langle A_1\rangle_{A_2*\mathbf{n}}\,\langle A_2\rangle_{\mathbf{n}}$ is in right normal form. Same argument applies to the left normal form. ☐

Lemma 3. *Let $P = \langle P_0\rangle_{\mathbf{n}}(P_1 \oplus \cdots \oplus P_k)$. Let A_i be the rightmost permutation braid in the right normal form of P_i for $i = 0, \ldots, k$. Then the rightmost permutation braid in the right normal form of P is $\langle A_0\rangle_{\mathbf{n}}(A_1 \oplus \cdots \oplus A_k)$.*

Proof. By Lemma 2, the rightmost permutation braid of $\langle P_0\rangle_{\mathbf{n}}$ is $\langle A_0\rangle_{\mathbf{n}}$. So the rightmost permutation braid of P equals the rightmost permutation braid of $\langle A_0\rangle_{\mathbf{n}}(P_1 \oplus \cdots \oplus P_k) = (P_{\theta^{-1}(1)} \oplus \cdots \oplus P_{\theta^{-1}(k)})\langle A_0\rangle_{\mathbf{n}}$, where θ is the induced permutation of A_0. Since $A_{\theta^{-1}(1)} \oplus \cdots \oplus A_{\theta^{-1}(k)}$ is the rightmost permutation braid of $P_{\theta^{-1}(1)} \oplus \cdots \oplus P_{\theta^{-1}(k)}$ and $(A_{\theta^{-1}(1)} \oplus \cdots \oplus A_{\theta^{-1}(k)})\langle A_0\rangle_{\mathbf{n}} = \langle A_0\rangle_{\mathbf{n}}(A_1 \oplus \cdots \oplus A_k)$ is a permutation braid, $\langle A_0\rangle_{\mathbf{n}}(A_1 \oplus \cdots \oplus A_k)$ is the rightmost permutation braid of P as desired. ☐

Corollary 4. *Let $\alpha = \langle\alpha_0\rangle_{\mathbf{n}}(\alpha_1 \oplus \cdots \oplus \alpha_k)$. Then $\inf(\alpha) = \min\{\inf(\alpha_i); i = 0 \text{ or } n_i > 1\}$ and $\sup(\alpha) = \max\{\sup(\alpha_i); i = 0 \text{ or } n_i > 1\}$.*

Figure 7. The normal form of a braid with an invariant standard curve

Proposition 5. *Let $\mathcal{C}$ be a standard curve system and $\alpha \in B_n$ such that $\alpha * \mathcal{C}$ is standard.*

(1) *Let $\alpha = \Delta^r A_1 \cdots A_l$ be a (left or right) normal form of α. Then $(A_i \cdots A_l) * \mathcal{C}$ is standard for $i = 1, \ldots, l$.*
(2) *If $\alpha = P^{-1}Q$ for some $P, Q \in B_n^+$ with $S(P) \cap S(Q) = \emptyset$, then $Q * \mathcal{C}$ is standard.*
(3) *If $\alpha = PQ^{-1}$ for some $P, Q \in B_n^+$ with $F(P) \cap F(Q) = \emptyset$, then $Q^{-1} * \mathcal{C}$ is standard.*

Proof. We may assume that $\mathcal{C}$ has only one component, because for $\beta \in B_n$, $\beta * \mathcal{C}$ is standard if and only if each component of $\beta * \mathcal{C}$ is standard. In particular, we assume that $\mathcal{C}$ is un-nested, that is, $\mathcal{C} = \mathcal{C}_{\mathbf{n}}$ for some partition $\mathbf{n}$ of n. Then (1) is immediate from Lemma 1 and Lemma 3. (2) and (3) follow from remark in the preliminary. □

Now we know that if a un-nested standard curve system is invariant under $\alpha \in B_n$, then we can see the standard curves in the normal form. For example, the braid in Figure 7 has an invariant standard curve system $\mathcal{C}_{\mathbf{n}}$, where $\mathbf{n} = (2, 1, 1)$.

Theorem 6. *There is an algorithm that decides, given an n-braid α, whether there exists a standard curve system that is invariant under α, and if there exists, finds one. The time complexity is $\mathcal{O}(n^3\ell^2)$, where ℓ is the word-length of α.*

Proof. Given a braid we can compute the right-canonical form $\alpha = \Delta^u A_\ell \cdots A_1$ in time $\mathcal{O}(n \log n\ell^2)$. There are $\binom{n}{2}$ candidates of standard curves invariant under α and for each candidate, we can check whether it is really invariant in time $\mathcal{O}(n\ell)$ as follows: Let θ_i be the induced permutation of A_i. Let $[a_0, b_0]$ be a set of consecutive integers. For $i = 1, \ldots, \ell$ do the followings

(1) Compute the set $S_i = \{\theta_i(a_0), \ldots, \theta_i(b_0)\}$.
(2) Let a_i and b_i be the minimum and maximum of the set S_i.
(3) If $b_i - a_i = b_0 - a_0$, proceed to $i + 1$. Otherwise, reject.
(4) Let $i = \ell + 1$. If u is even, *accept* if $a_0 = a_\ell$ and $b_0 = b_\ell$. Otherwise, *reject*. If u is odd, *accept* if $a_0 = n + 1 - b_\ell$ and $b_0 = n + 1 - a_\ell$. Otherwise, *reject*.

Since the steps (1), (2) and (4) can be done in time linear to n and (3) is repetition of (2) ℓ times, we are done. □

FIGURE 8

Proposition 7. *Let $\mathcal{C}$ be an essential curve system in D_n (in general, non-standard). If $P_i * \mathcal{C}$ is standard for positive braids P_i, $i = 1, 2$, then so are $(P_1 \wedge_R P_2) * \mathcal{C}$ and $(P_1 \vee_L P_2) * \mathcal{C}$.*

Proof. Let $R = P_1 \wedge_R P_2$ and $P_i = Q_i R$ for $i = 1, 2$. Let $\mathcal{C}_i = P_i * C = Q_i R * \mathcal{C}$, $i = 1, 2$. Since $Q_1^{-1} * \mathcal{C}_1 = R * \mathcal{C} = Q_2^{-1} * \mathcal{C}_2$, $(Q_1 Q_2^{-1}) * \mathcal{C}_2 = \mathcal{C}_1$. So $Q_2^{-1} * \mathcal{C}_2 = R * \mathcal{C}$ is standard.

Now let $T = P_1 \vee_L P_2 = S_1 P_1 = S_2 P_2$. Since $T * C = (S_i P_i) * \mathcal{C} = S_i * \mathcal{C}_i$, for $i = 1, 2$, $S_1 * \mathcal{C}_1 = S_2 * \mathcal{C}_2$. Now $(S_1^{-1} S_2) * \mathcal{C}_2 = \mathcal{C}_1$ and so $S_2 * \mathcal{C}_2 = T * \mathcal{C}$ is standard. □

The following corollary answers the question at the beginning.

Corollary 8. *Given any curve system $\mathcal{C}$, there exist a unique positive braid P such that $P * \mathcal{C}$ is standard and if Q is another positive braid with $Q * \mathcal{C}$ standard, then $Q = RP$ for some positive braid R.*

We close this note with an example. Let $\mathcal{C}$ be an essential curve system such that $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. Let P, P_1 and P_2 be shortest positive braids that standardize the curve systems $\mathcal{C}$, $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively. In this situation we can ask whether P can be computed directly from P_1 and P_2. Because $P \succcurlyeq P_1$ and $P \succcurlyeq P_2$, the natural candidate is $P = P_1 \vee_L P_2$, that is the shortest positive braid with the property $P \succcurlyeq P_1$ and $P \succcurlyeq P_2$. But it is not true in general. Consider the curve systems $\mathcal{C} = C_1 \cup C_2$ as Figure 9. Then standardizing braid of C_1 and C_2 are σ_1 and σ_3, respectively. But the standardizing braid of $\mathcal{C}$ is $\sigma_2\sigma_1\sigma_3$. Note that $\sigma_2\sigma_1\sigma_3 \succcurlyeq \sigma_1 \vee_L \sigma_3 (= \sigma_1\sigma_3)$ but the two braids are not equal.

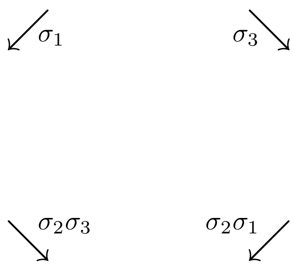

FIGURE 9. Standardization of a curve system

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DEPARTMENT OF MATHEMATICS, KONKUK UNIVERSITY, 1 HWAYANG-DONG GWANGJIN-GU, SEOUL 143-701, KOREA

E-mail address: sangjin@konkuk.ac.kr