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COMPARISON BETWEEN THE TEICHMÜLLER SPACE AND THE GOLDMAN SPACE

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ABSTRACT. In this article we compare some results of the Teichmüller space $\mathfrak{T}(M)$ and the Goldman space $\mathcal{G}(M)$ on a smooth surface M. We present an algebraic expression of an isometric embedding of $\mathfrak{T}(M)$ into $\mathcal{G}(M)$ and show that the modified Goldman's length parameters on $\mathcal{G}(M)$ is an isometric extension of the Fenchel-Nielsen's length parameter on $\mathfrak{T}(M)$.

1. INTRODUCTION

Let M be a connected smooth surface. The Teichmüller space $\mathfrak{T}(M)$ on M is a deformation space of hyperbolic structures on M and the Goldman space $\mathcal{G}(M)$ is that of convex real projective structures on M. The study of real projective structures has been quite active. Ehresmann, Kuiper, Benzécri, Kobayashi, and Thurston have done important work. Recently Goldman and Choi lead this field. Choi and Goldman [4] showed $\mathcal{G}(M)$ is a component of the deformation space of real projective structures $\mathbb{RP}^2(M)$ and contains the Teichmüller space $\mathfrak{T}(M)$. In this paper we survey some theorems about (G,X)-structures on M and present an algebraic expression of an isometric embedding of $\mathfrak{T}(M)$ into $\mathcal{G}(M)$. Finally we show that the modified Goldman's length parameters on $\mathcal{G}(M)$ is an isometric extension of the Fenchel-Nielsen's length parameters on $\mathfrak{T}(M)$.

2. DEFORMATION SPACE OF (G,X)-STRUCTURES

2.1. (G,X)**-structures.** An action of a connected Lie group G on a smooth n-manifold X is called *strongly effective* if $g_1, g_2 \in G$ agree on a nonempty open set of X, then $g_1 = g_2$. Let Ω be an open subset of X. A map $\phi : \Omega \to X$ is called *locally*-(G,X) if for each component $W \subset \Omega$, there exists a (G,X)-transformation $g \in G$ such that $\phi|_W = g|_W$.

A (G,X)-*structure* on a connected smooth n-manifold M is a maximal collection of coordinate charts $\{(U_\alpha, \psi_\alpha)\}$ such that

(1) G acts on X strongly effectively.
(2) $\{U_\alpha\}$ is an open covering of M.

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(3) For each α, $\psi_\alpha : U_\alpha \to X$ is a diffeomorphism onto its image.
(4) If (U_α, ψ_α) and (U_β, ψ_β) are two coordinate charts with $U_\alpha \cap U_\beta \neq \emptyset$, then the transition function $\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(U_\alpha \cap U_\beta) \to \psi_\beta(U_\alpha \cap U_\beta)$ is locally-(G, X).

Now we give two examples of (G, X)-structures.

Example 2.1. Let $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the upper half complex plane. Then $\mathbf{SL}(2, \mathbb{R})$ acts on $\mathbb{H}^2$ by

$$A \cdot z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d} \,. \tag{2.1}$$

Since we have $A \cdot z = (-A) \cdot z$ for any $A \in \mathbf{SL}(2, \mathbb{R})$ and $z \in \mathbb{H}^2$, the Lie group $\mathbf{PSL}(2, \mathbb{R}) = \mathbf{SL}(2, \mathbb{R})/\pm I$ acts strongly effectively on $\mathbb{H}^2$.

Example 2.2. Let $\mathbb{RP}^2$ be the space of all lines through the origin in $\mathbb{R}^3$. For a nonzero vector v in $\mathbb{R}^3$, $[v]$ denotes the corresponding point in $\mathbb{RP}^2$. Let B be an element of $\mathbf{GL}(3, \mathbb{R})$. Then B preserves lines through the origin and induces a projective transformation of $\mathbb{RP}^2$. Thus $\mathbf{GL}(3, \mathbb{R})$ acts on $\mathbb{RP}^2$ by

$$B \cdot [v] = [Bv]. \tag{2.2}$$

Since the scalar matrices $\mathbb{R}^* \subset \mathbf{GL}(3, \mathbb{R})$ acts trivially on $\mathbb{RP}^2$, the Lie group $\mathbf{PGL}(3, \mathbb{R}) = \mathbf{GL}(3, \mathbb{R})/\mathbb{R}^*$ acts strongly effectively on $\mathbb{RP}^2$.

Definition 2.3. The $\big(\mathbf{PSL}(2, \mathbb{R}), \mathbb{H}^2\big)$-structures and $\big(\mathbf{PGL}(3, \mathbb{R}), \mathbb{RP}^2\big)$-structures on a smooth surface M are called the *hyperbolic structures* and *real projective structures* on M respectively.

2.2. **Developing pair.** A manifold equipped with a (G, X)-structure is called a (G, X)-*manifold.* Let N be a (G, X)-manifold. If $f : M \to N$ is a local diffeomorphism of smooth manifolds, then we can give the induced (G, X)-structure on M via f. In particular every covering space of a (G, X)-manifold has the canonically induced (G, X)-structure.

Let M and N be (G, X)-manifolds and $f : M \to N$ a smooth map. Then f is called a (G, X)-*map* if for each coordinate chart (U, ψ_U) on M and (V, ψ_V) on N, the composition $\psi_V \circ f \circ \psi_U^{-1} : \psi_U(f^{-1}(V) \cap U) \to \psi_V(f(U) \cap V)$ is locally-(G, X). The following famous theorem is due to Ehresmann [5] and Thurston [15].

Theorem 2.4. *Let M be a (G, X)-manifold and $p : \tilde{M} \to M$ denotes a fixed universal covering of M. Let π be the corresponding group of covering transformations.*

(1) *There exist a (G, X)-map $\mathbf{dev} : \tilde{M} \to X$ and a homomorphism $h : \pi \to G$ such that for each $\gamma \in \pi$ the following diagram commutes.*

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\mathbf{dev}} & X \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle h(\gamma)} \\ \tilde{M} & \xrightarrow[\mathbf{dev}]{} & X \end{array}$$

(2) *Suppose* $(\mathbf{dev}', h')$ *is another pair satisfying above condition, then there exists* $g \in G$ *such that* $\mathbf{dev}' = g \circ \mathbf{dev}$ *and* $h' = \iota_g \circ h$ *where* $\iota_g : G \to G$ *denotes the inner automorphism defined by* g*; i.e.* $h'(\gamma) = (\iota_g \circ h)(\gamma) = g \circ h(\gamma) \circ g^{-1}$.

$$\begin{array}{ccccc} \tilde{M} & \xrightarrow{\mathbf{dev}} & X & \xrightarrow{g} & X \\ \gamma\downarrow & & \downarrow h(\gamma) & & \downarrow h'(\gamma) \\ \tilde{M} & \xrightarrow[\mathbf{dev}]{} & X & \xrightarrow[g]{} & X \end{array}$$

The (G, X)-map $\mathbf{dev} : \tilde{M} \to X$ is called a *developing map*, the homomorphism $h : \pi \to G$ is called the *holonomy homomorphism*, the image $\Omega = \mathbf{dev}(\tilde{M}) \subset X$ is called the *developing image,* and the image $\Gamma = h(\pi) \subset G$ is called the *holonomy group.* Thus the *developing pair* $(\mathbf{dev}, h)$ is unique up to the G-action by composition and conjugation respectively.

2.3. **Deformation space.** Consider a pair (f, N) where N is a (G, X)-manifold and $f : M \to N$ is a diffeomorphism. Then M admits the induced (G, X)-structure via f. The set of all such pairs (f, N) is denoted by $\mathcal{A}(M)$. Then $\mathcal{A}(M)$ is the space of all (G, X)-structures on M. We say two pairs (f', N') and (f, N) in $\mathcal{A}(M)$ are *equivalent* if there exists a (G, X)-diffeomorphism $g' : N' \to N$ such that $g' \circ f'$ is isotopic to f ; that is, there exists a diffeomorphism $g : M \to M$, which is isotopic to the identity map I_M such that the following diagram commutes :

$$\begin{array}{ccc} M & \xrightarrow{f'} & N' \\ g\downarrow & & \downarrow g' \\ M & \xrightarrow[f]{} & N \end{array}$$

The set of equivalence classes $\mathcal{A}(M)/\sim$ will be denoted by $\mathfrak{D}(M)$ and called the *deformation space* of (G, X)-structures on M.

The deformation space $\mathfrak{D}(M)$ has the natural topology. Let $\mathrm{Diff}(M)^0$ be the space of all diffeomorphisms of M which are isotopic to the identity map I_M. Then we may think the deformation space $\mathfrak{D}(M)$ consists of diffeomorphisms $f : M \to N$ modulo the action of $\mathrm{Diff}(M)^0$ given by

$$g : f \mapsto f \circ g$$

where $g \in \mathrm{Diff}(M)^0$. Give $\mathfrak{D}(M)$ the quotient topology induced from the C^∞-topology on the space of diffeomorphisms $f : M \to N$.

Definition 2.5. Let M be a connected smooth surface. The *Teichmüller space* $\mathfrak{T}(M)$ is the deformation space of hyperbolic structures on M. The deformation space of real projective structures on M will be denoted by $\mathbb{RP}^2(M)$.

2.4. **The orbit space** $\mathrm{Hom}(\pi, G)/G$**.** The deformation $\mathfrak{D}(M)$ is closely related to $\mathrm{Hom}(\pi, G)/G$ the orbit space of homomorphisms $\phi : \pi \to G$. Let M be a compact connected smooth manifold. Since M is compact, the fundamental group π of M admits finite generators $\gamma_1, \cdots, \gamma_m$ with finite relations $R_1, \cdots, R_k$. For example if M

is $\Sigma(g,n)$, that is a compact connected smooth surface with g-genus and n-boundary components, then π admits $2g+n$ generators $A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_n$ with a single relation

$$R = A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1} C_1 \cdots C_n.$$

From the correspondence of the homomorphism $\phi : \pi \to G$ to the image of generators $g_1 = \phi(\gamma_1), \cdots, g_m = \phi(\gamma_m)$, $\mathrm{Hom}(\pi, G)$ may be identified with the collection of all m-tuples $(g_1, \ldots, g_m) \subset G^m$ satisfying

$$R_1(g_1, \ldots, g_m) = I, \ \cdots, \ R_k(g_1, \ldots, g_m) = I.$$

The group G acts on $\mathrm{Hom}(\pi, G)$ by conjugation as follow ; For $g \in G$ and $\phi \in \mathrm{Hom}(\pi, G)$, the action $g \cdot \phi$ is defined by

$$(g \cdot \phi)(\gamma) = g \circ \phi(\gamma) \circ g^{-1}$$

where $\gamma \in \pi$. Taking the holonomy homomorphism of a (G, X)-structure defines a map

$$\mathbf{hol} : \mathfrak{D}(M) \longrightarrow \mathrm{Hom}(\pi, G)/G$$

which is a local diffeomorphism. See Goldman [6] and Johnson [10] for details.

3. The Teichmüller space vs. the Goldman space

3.1. **Comparison between the Teichmüller space and the Goldman space.** Let M be a hyperbolic surface. Then the developing map $\mathbf{dev}$ is a diffeomorphism from $\tilde{M}$ onto a convex domain $\Omega = \mathbf{dev}(\tilde{M}) \subset \mathbb{H}^2$ and the holonomy homomorphism h is an isomorphism from π onto a discrete subgroup $\Gamma = h(\pi) \subset \mathbf{PSL}(2, \mathbb{R})$ which acts properly and freely on Ω. Thus if a compact connected smooth surface M has a hyperbolic structure, the M is diffeomorphic to the quotient Ω/Γ.

A domain $\Omega \subset \mathbb{RP}^2$ is called *convex* if there exist a projective line $\ell \subset \mathbb{RP}^2$ such that $\Omega \subset (\mathbb{RP}^2 - \ell)$ and Ω is a convex subset of the affine plane $\mathbb{RP}^2 - \ell$; that is, if $x, y \in \Omega$, then the line segment $\overline{xy}$ lies in Ω. By definition, $\mathbb{RP}^2$ itself is not convex.

Suppose M is a real projective surface. Then generally the developing map is just a local diffeomorphism and the developing image may be not convex. We can find such examples in Sullivan and Thurston's paper [14].

Definition 3.1. A real projective structure on M is called *convex* if the developing map $\mathbf{dev} : \tilde{M} \to \mathbb{RP}^2$ is a diffeomorphism onto a convex domain in $\mathbb{RP}^2$.

The following fundamental theorem is from Goldman's paper [7].

Theorem 3.2. *Let M be a real projective surface. Then the following statements are equivalent.*

1. *M has a convex real projective structure.*
2. *M is projectively diffeomorphic to a quotient Ω/Γ where $\Omega \subset \mathbb{RP}^2$ is a convex domain and $\Gamma \subset \mathbf{PGL}(3, \mathbb{R})$ is a discrete group acting properly and freely on Ω.*

Definition 3.3. The *Goldman space* $\mathcal{G}(M)$ is the subset of $\mathbb{RP}^2(M)$ consisting of the equivalence classes of convex real projective structures.

The Goldman space $\mathcal{G}(M)$ is an analogue of the Teichmüller space $\mathfrak{T}(M)$. Let $M = \Sigma(g, n)$ be a compact oriented surface with g-genus, n-boundary components. The following results are from Atiyah and Bott [1], Audin [2], Goldman [7], [8], and Guruprasad, Huebschmann, Jeffrey, Weinstein [9].

Theorem 3.4. *Let $M = \Sigma(g, n)$ with $\chi(M) = 2 - 2g - n < 0$. Then*

(1) $\mathbf{hol} : \mathfrak{T}(M) \to \mathrm{Hom}(\pi, \mathbf{PSL}(2,\mathbb{R}))/\mathbf{PSL}(2,\mathbb{R})$ *is an embedding onto open subset* $\mathrm{Hom}(\pi, \mathbf{PSL}(2,\mathbb{R}))^s/\mathbf{PSL}(2,\mathbb{R})$.
(2) $\mathrm{Hom}(\pi, \mathbf{PSL}(2,\mathbb{R}))^s/\mathbf{PSL}(2,\mathbb{R})$ *is a real analytic manifold of dimension* $-3 \cdot \chi(M) = 6g - 6 + 3n$.
(3) $\mathfrak{T}(M)$ *is diffeomorphic to* $\mathbb{R}^{-3\cdot\chi(M)} = \mathbb{R}^{6g-6+3n}$.
(4) *If M has boundary (i.e. $n > 0$), then $\mathfrak{T}(M)$ is a Poisson manifold.*
(5) *If M is closed (i.e. $n = 0$), then $\mathfrak{T}(M)$ is a Kähler manifold.*

Theorem 3.5. *Let $M = \Sigma(g, n)$ with $\chi(M) = 2 - 2g - n < 0$. Then*

(1) $\mathbf{hol} : \mathcal{G}(M) \to \mathrm{Hom}(\pi, \mathbf{PGL}(3,\mathbb{R}))/\mathbf{PGL}(3,\mathbb{R})$ *is an embedding onto open subset* $\mathrm{Hom}(\pi, \mathbf{PGL}(3,\mathbb{R}))^s/\mathbf{PGL}(3,\mathbb{R})$.
(2) $\mathrm{Hom}(\pi, \mathbf{PGL}(3,\mathbb{R}))^s/\mathbf{PGL}(3,\mathbb{R})$ *is a real analytic manifold of dimension* $-8 \cdot \chi(M) = 16g - 16 + 8n$.
(3) $\mathcal{G}(M)$ *is diffeomorphic to* $\mathbb{R}^{-8\cdot\chi(M)} = \mathbb{R}^{16g-16+8n}$.
(4) *If M has boundary (i.e. $n > 0$), then $\mathcal{G}(M)$ is a Poisson manifold.*
(5) *If M is closed (i.e. $n = 0$), then $\mathcal{G}(M)$ is a symplectic manifold.*

Conjecture 3.6 (Goldman)**.** *If M is closed, then $\mathcal{G}(M)$ is a Kähler manifold.*

Theorem 3.7 (Choi, Goldman [4])**.** *The Goldman space $\mathcal{G}(M)$ is a component of $\mathbb{RP}^2(M)$ which contains the Teichmüller space $\mathfrak{T}(M)$.*

The element of the Teichmüller space $\mathfrak{T}(M)$ will be identified with a conjugacy class of $\mathrm{Hom}(\pi, \mathbf{PSL}(2,\mathbb{R}))$. Similarly the element of the Goldman space $\mathcal{G}(M)$ will be identified with a conjugacy class of $\mathrm{Hom}(\pi, \mathbf{PGL}(3,\mathbb{R}))$. Since $\mathfrak{T}(M)$ embeds into $\mathcal{G}(M)$, we shall explicitly calculate the formula of the algebraic presentation of an embedding of $\mathfrak{T}(M)$ into $\mathcal{G}(M)$.

3.2. **Positive hyperbolic elements.** An element A of $\mathbf{SL}(2,\mathbb{R})$ is said to be *hyperbolic* if A has two distinct real eigenvalues. Since the characteristic polynomial of A is $f(\lambda) = \lambda^2 - t\lambda + 1$ where $t = \mathrm{tr}(A)$, A is hyperbolic if and only if $\mathrm{tr}(A)^2 > 4$.

Let $A \in \mathbf{PSL}(2,\mathbb{R})$. Since the absolute value of trace is still defined, A is said to be *hyperbolic* if $|\mathrm{tr}(A)| > 2$. Thus A can be expressed by the diagonal matrix

$$\pm \begin{bmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{bmatrix} \tag{3.1}$$

via an $\mathbf{SL}(2,\mathbb{R})$-conjugation where $0 < \alpha^{-1} < 1 < \alpha$.

The homomorphism $\mathbf{GL}(3,\mathbb{R}) \to \mathbf{SL}(3,\mathbb{R})$ defined by $A \mapsto (\det A)^{-1/3}A$ induces an isomorphism $\mathbf{PGL}(3,\mathbb{R}) = \mathbf{GL}(3,\mathbb{R})/\mathbb{R}^* \to \mathbf{SL}(3,\mathbb{R})$. Thus from now on we shall identify the groups $\mathbf{PGL}(3,\mathbb{R})$ and $\mathbf{SL}(3,\mathbb{R})$.

An element $A \in \mathbf{SL}(3,\mathbb{R})$ is called *positive hyperbolic* if it has three distinct positive real eigenvalues. If A is positive hyperbolic, then it can be represented by the diagonal matrix

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix} \tag{3.2}$$

via an $\mathbf{SL}(3,\mathbb{R})$-conjugation where $\lambda\,\mu\,\nu = 1$ and $0 < \lambda < \mu < \nu$.

The following theorem is one of the analogues between hyperbolic structures and real projective structures proved by Kuiper [13].

Theorem 3.8. *Let $M = \Sigma(g, n)$ with $\chi(M) < 0$.*

(1) *If M is a hyperbolic surface, then every nontrivial element of holonomy group $\Gamma \subset \mathbf{PSL}(2,\mathbb{R})$ is hyperbolic.*
(2) *If M is a convex real projective surface, then every nontrivial element of holonomy group $\Gamma \subset \mathbf{SL}(3,\mathbb{R})$ is positive hyperbolic.*
(3) *Either the boundary of the developing image is a conic in $\mathbb{RP}^2$ or is not $C^{1+\varepsilon}$ for any $\varepsilon > 0$.*

It is known that the boundary $\partial\Omega$ is a conic if and only if the convex real projective structure on M arises from a hyperbolic structure on M. Let Ω be the domain in $\mathbb{RP}^2$ defined by

$$\Omega = \{[x, y, s] \in \mathbb{RP}^2 \mid x^2 + y^2 - s^2 < 0\}. \tag{3.3}$$

Then Ω has a conic boundary $\partial\Omega$. Let M be a surface with a hyperbolic structure. Composing the developing map $\tilde{M} \to \mathbb{H}^2$ with an isometry $\mathbb{H}^2 \to \Omega \subset \mathbb{RP}^2$ and the holonomy homomorphism $\pi \to \mathbf{PSL}(2,\mathbb{R})$ with an embedding $\mathbf{PSL}(2,\mathbb{R}) \to \mathbf{SL}(3,\mathbb{R})$ realizes M as a convex real projective surface. The goal of next section is to define an isometry $\mathbb{H}^2 \to \Omega$ and an embedding $\mathbf{PSL}(2,\mathbb{R}) \to \mathbf{SL}(3,\mathbb{R})$.

4. Embedding of the Teichmüller space into the Goldman space

4.1. The Hilbert metric. To define an isometry $\mathbb{H}^2 \to \Omega$, we need a knowledge about the Hilbert metric. Hilbert discovered a metric on a bounded convex domain Ω in $\mathbb{R}^2$ (or equivalently in $\mathbb{C}$). For more detail, see Kobayashi's paper [12].

Let z_1, z_2, z_3, z_4 be four distinct points in the extended complex numbers $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The *cross-ratio* of four distinct points in $\bar{\mathbb{C}}$ is defined by

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_4)}. \tag{4.1}$$

Remark 4.1. There are six different methods to define the cross-ratio on $\bar{\mathbb{C}}$. We adopt this presentation since it is an easy way to understand the Hilbert metric.

A convex domain (may be unbounded) which does not contain any full straight line is called *proper*. A *strictly* convex domain is a convex domain whose boundary $\partial\Omega$ does not contain any line segment.

Definition 4.2. Let Ω be a properly and strictly convex domain in $\mathbb{C}$. For a distinct points z, w in Ω, the *Hilbert distance* between z and w is defined by

$$d_H(z,w) = \log\,[z^*, z, w, w^*] \tag{4.2}$$

where z^*, w^* are the boundary points in $\partial\Omega$ which lie on the straight line joining z and w such that z is between z^* and w.

If we add $d_H(z,z) = 0$, then the Hilbert distance d_H defines a complete metric on Ω called the *Hilbert metric.* The Hilbert metric d_H has the following properties:

(1) There is a unique geodesic between two points in Ω or $\partial\Omega$.
(2) The geodesics are straight lines in Euclidean sense.

Kuiper [13] showed that the developing images $\Omega \subset \mathbb{RP}^2$ of convex real projective structures are properly and strictly convex domains. This yields that every developing image of a convex real projective structure has the Hilbert metric.

4.2. **The Poincaré metric.** Let $\mathbb{H}_P = (\mathbb{H}^2, d_P)$ be the upper half complex plane $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ with the Poincaré metric d_P ; i.e. the lines in $\mathbb{H}_P$ are the semi-circles centered at the x-axis and the rays orthogonal to the x-axis. The Poincaré metric on $\mathbb{H}_P$ is defined by

$$d_P(z,w) = \log\,[z', z, w, w'] \tag{4.3}$$

where z', w' are the boundary points in the extended x-axis $\bar{\mathbb{R}} = \mathbb{R} \cup \infty$ which lie on the line joining z and w such that z is between z' and w.

The elements of $\mathbf{PSL}(2,\mathbb{R})$ act on $\mathbb{H}_P$ as the linear fractional transformations in (2.1). Since the linear fractional transformations on $\bar{\mathbb{C}}$ preserve the cross-ratio, we have the following theorem.

Theorem 4.3. *The group $\mathcal{I}_1$ of isometries of the upper half plane $\mathbb{H}_P$ is*

$$\mathcal{I}_1 = \mathbf{PSL}(2,\mathbb{R}) = \left\{ \pm \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a\,d - b\,c = 1 \right\}. \tag{4.4}$$

4.3. **The Poincaré disc.** Let $\mathbb{D}_P = (\mathbb{D}^2, d_P)$ be the unit disc $\mathbb{D}^2 = \{z \in \mathbb{C} \mid |z| < 1\}$ with the Poincaré metric d_P ; i.e. the lines in D_P are the arcs of circles which are orthogonal the boundary of $\mathbb{D}^2$ and the segments through the origin. The Poincaré metric on $\mathbb{D}_P$ is defined similarly as in (4.3).

Theorem 4.4. *The linear fractional transformation $G_1 : \mathbb{H}_P \to \mathbb{D}_P$ defined by*

$$w = G_1(z) = \frac{z - i}{-iz + 1} \tag{4.5}$$

is an isometry with the inverse $G_1^{-1} \overset{\text{let}}{=} F_1 : \mathbb{D}_P \to \mathbb{H}_P$

$$z = F_1(w) = \frac{w + i}{iw + 1}. \tag{4.6}$$

Remark 4.5. $G_1 : \mathbb{H}_P \to \mathbb{D}_P$ maps $\{-1, 0, 1, \infty, i\}$ to $\{-1, -i, 1, i, 0\}$ respectively.

FIGURE 1. The Poincaré and the Hilbert metric on $\mathbb{D}^2$

4.4. **Relation between the Poincaré metric and the Hilbert metric.** Let $\mathbb{D}_H = (\mathbb{D}^2, d_H)$ be the unit disc $\mathbb{D}^2$ with the Hilbert metric d_H ; i.e. the lines in $\mathbb{D}_H$ are the Euclidean line segments.

Let $\Sigma = \{(x, y, s) \in \mathbb{R}^3 \mid x^2 + y^2 + s^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$ with the north pole $n = (0, 0, 1)$. The stereographic projection $P : \Sigma - \{n\} \to \mathbb{R}^2$ defined by

$$P(x, y, s) = \left(\frac{x}{1-s}, \frac{y}{1-s} \right)$$

is a conformal diffeomorphism with the inverse

$$P^{-1}(x, y) = \left(\frac{2\,x}{1 + x^2 + y^2}, \frac{2\,y}{1 + x^2 + y^2}, \frac{-1 + x^2 + y^2}{1 + x^2 + y^2} \right).$$

FIGURE 2. Isometry $G_2 = p_{xy} \circ P^{-1} : \mathbb{D}_P \to \mathbb{D}_{H'}$

Let $G_2 : \mathbb{D}_P \to \mathbb{D}_H$ be a mapping defined by $G_2 = p_{xy} \circ P^{-1}$ where $p_{xy} : \mathbb{R}^3 \to \mathbb{R}^2$ is the projection to the xy-plain ; i.e.

$$G_2(x, y) = \left(\frac{2\,x}{1 + x^2 + y^2}, \frac{2\,y}{1 + x^2 + y^2} \right). \tag{4.7}$$

Then G_2 is a diffeomorphism with the inverse $G_2^{-1} \overset{\text{let}}{=} F_2 : \mathbb{D}_H \to \mathbb{D}_P$

$$F_2(x,y) = \left(\frac{x}{1+\sqrt{1-x^2-y^2}}, \frac{y}{1+\sqrt{1-x^2-y^2}} \right). \tag{4.8}$$

In the complex variables $w \in \mathbb{D}_P,\ z \in \mathbb{D}_H$, the mappings G_2, F_2 are represented by

$$z = G_2(w) = \frac{2\,w}{1+|w|^2}, \qquad w = F_2(z) = \frac{z}{1+\sqrt{1-|z|^2}}. \tag{4.9}$$

Unfortunately the mapping $G_2 : \mathbb{D}_P \to \mathbb{D}_H$ is not an isometry. Through a little modification of the Hilbert metric, we can show that $G_2 : \mathbb{D}_P \to \mathbb{D}_{H'}$ is an isometry. Let $\mathbb{D}_{H'} = (\mathbb{D}^2, d_{H'})$ be the unit disc $\mathbb{D}^2$ with the modified Hilbert metric $d_{H'}$ defined by

$$d_{H'}(z,w) = \frac{1}{2}\, d_H(z,w). \tag{4.10}$$

Theorem 4.6. *The mapping $G_2 : \mathbb{D}_P \to \mathbb{D}_{H'}$ is an isometry.*

Sketch of proof. For $w_1, w_2 \in \mathbb{D}_P$, let $z_j = G_2(w_j) \in \mathbb{D}_{H'}$. Then we can compute

$$[w_1', w_1, w_2, w_2'] = [z_1^*, z_1, z_2, z_2^*]^{\frac{1}{2}}. \tag{4.11}$$

Since $d_P(w_1,w_2) = \log [w_1', w_1, w_2, w_2']$ and $d_{H'}(z_1,z_2) = \frac{1}{2}\log [z_1^*, z_1, z_2, z_2^*]$, the mapping $G_2 : \mathbb{D}_P \to \mathbb{D}_{H'}$ is an isometry. □

FIGURE 3. The image of lines through $G_2 : \mathbb{D}_P \to \mathbb{D}_{H'}$

4.5. The Hilbert disk $\mathbb{D}_{H'}$ and the convex domain $\Omega_{H'}$ in $\mathbb{RP}^2$.

Let $\Omega_{H'} = (\Omega, d_{H'})$ be the convex domain $\Omega \subset \mathbb{RP}^2$ defined by

$$\Omega = \{[x,y,s] \in \mathbb{RP}^2 \mid x^2+y^2-s^2 < 0\}$$

with the modified Hilbert metric $d_{H'} = \frac{1}{2} d_H$. Then $\Omega_{H'}$ is a properly and strictly convex domain with the conic boundary.

Theorem 4.7. *The mapping $G_3 : \mathbb{D}_{H'} \to \Omega_{H'}$ defined by*

$$G_3(z) = G_3(x+i\,y) = [\,x, y, 1\,] \tag{4.12}$$

is an isometry with the inverse $G_3^{-1} \overset{\text{let}}{=} F_3 : \Omega_{H'} \to \mathbb{D}_{H'}$

$$F_3\left([x, y, s]\right) = \frac{x}{s} + i\,\frac{y}{s}. \tag{4.13}$$

The goal of this section is to define an isometry $\mathbb{H}^2 \to \Omega$.

Theorem 4.8 (Kim [11]). *The mapping* $G : \mathbb{H}_P \to \Omega_{H'}$ *defined by the compositions* $G = G_3 \circ G_2 \circ G_1$ *is an isometry such that*

$$G(z) = G(x + i\,y) = [\,2x,\ x^2 + y^2 - 1,\ x^2 + y^2 + 1\,] \tag{4.14}$$

with the inverse $G^{-1} \overset{\text{let}}{=} F = F_1 \circ F_2 \circ F_3$

$$F([x, y, s]) = \left(\frac{x}{s - y}\right) + i\left(\frac{s}{s - y}\right)\sqrt{1 - \frac{x^2}{s^2} - \frac{y^2}{s^2}}. \tag{4.15}$$

4.6. **Embedding of $\mathbf{PSL}(2, \mathbb{R})$ into $\mathbf{SL}(3, \mathbb{R})$.** To realizes a hyperbolic structure on M as a convex real projective structure, consider the final goal of this section, which is to define an embedding $\mathbf{PSL}(2, \mathbb{R}) \to \mathbf{SL}(3, \mathbb{R})$.

Theorem 4.9 (Kim [11]). *The mapping* $\varphi : \mathbf{PSL}(2, \mathbb{R}) \to \mathbf{SL}(3, \mathbb{R})$ *defined by*

$$\varphi(A) = \begin{bmatrix} ad + bc & ac - bd & ac + bd \\ ab - cd & \frac{a^2 - b^2 - c^2 + d^2}{2} & \frac{a^2 + b^2 - c^2 - d^2}{2} \\ ab + cd & \frac{a^2 - b^2 + c^2 - d^2}{2} & \frac{a^2 + b^2 + c^2 + d^2}{2} \end{bmatrix} \quad \text{for } A = \pm \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{4.16}$$

is an embedding of $\mathbf{PSL}(2, \mathbb{R})$ *into* $\mathbf{SL}(3, \mathbb{R})$.

Sketch of proof. Since the mapping $G : \mathbb{H}_P \to \Omega_{H'}$ is an isometry with the inverse F, we can define a mapping $\varphi : \mathbf{PSL}(2, \mathbb{R}) \to \mathbf{SL}(3, \mathbb{R})$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{H}_P & \xrightarrow{\ G\ } & \Omega_{H'} \\ {\scriptstyle A}\big\downarrow & & \big\downarrow{\scriptstyle \varphi(A)} \\ \mathbb{H}_P & \xrightarrow[\ G\ ]{} & \Omega_{H'} \end{array}$$

Let $[x, y, s] = [z, s]$ be a point in $\Omega_{H'} \subset \mathbb{RP}^2$ and $A \in \mathbf{PSL}(2, \mathbb{R})$ the matrix representation of an isometry f_1 of H_P. By Theorems 4.3 and 4.8,

$$(G \circ f_1 \circ F)[\,x, y, s\,] = \left[(\alpha^2 z + \beta^2 \bar{z} + 2\alpha\beta s), (2\,\mathrm{Re}(\alpha\bar{\beta} z) + |\alpha|^2 s + |\beta|^2 s)\right] \tag{4.17}$$

where $\alpha = (\frac{a+d}{2}) + i(\frac{b-c}{2})$ and $\beta = (\frac{b+c}{2}) + i(\frac{a-d}{2})$. To describe the matrix representation of $(G \circ f_1 \circ F)$, plug in the standard basis $[1, 0, 0], [0, 1, 0], [0, 0, 1]$ to the linear transformation $(G \circ f_1 \circ F)$. Then we obtain the matrix representation $\varphi(A)$ as in (4.16). Suppose $\varphi(A_1) = \varphi(A_2)$ for $A_1, A_2 \in \mathbf{SL}(2, \mathbb{R})$. Then we can show $A_1 = A_2$ or $A_1 = -A_2$. We can also compute $\varphi(A_1 A_2) = \varphi(A_1)\varphi(A_2)$. Therefore $\varphi : \mathbf{PSL}(2, \mathbb{R}) \to \mathbf{SL}(3, \mathbb{R})$ is an embedding. □

After some the Mathematica computations, we have the following properties.

Proposition 4.10. *Let* $B = \varphi(A) \in \mathbf{SL}(3, \mathbb{R})$ *for* $A \in \mathbf{PSL}(2, \mathbb{R})$. *Then,*

(1) $B \in \mathbf{SO}(2, 1) \subset \mathbf{SL}(3, \mathbb{R})$.

(2) *Suppose* $\pm\alpha$, $\pm\alpha^{-1}$ *are the eigenvalues of* A. *Then* $\alpha^2, 1, \alpha^{-2}$ *are the eigenvalues of* B.

Therefore if $A \in \mathbf{PSL}(2,\mathbb{R})$ is a hyperbolic element, then $B = \varphi(A) \in \mathbf{SL}(3,\mathbb{R})$ is a positive hyperbolic element. We can derive that the hyperbolic structures embeds into convex real projective structures through the identification of the conjugacy classes of $[\mathbf{PSL}(2,\mathbb{R})] \hookrightarrow [\mathbf{SL}(3,\mathbb{R})]$

$$\pm\begin{bmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{bmatrix} \hookrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\alpha^{-2}+\alpha^2}{2} & \frac{\alpha^{-2}-\alpha^2}{2} \\ 0 & \frac{\alpha^{-2}-\alpha^2}{2} & \frac{\alpha^{-2}+\alpha^2}{2} \end{bmatrix} \simeq \begin{bmatrix} \alpha^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha^2 \end{bmatrix}. \tag{4.18}$$

5. Fenchel-Nielsen's and Goldman's length parameters

An element $B \in \mathbf{SL}(3,\mathbb{R})$ is called positive hyperbolic if it has three distinct positive real eigenvalues. The set of positive hyperbolic elements of $\mathbf{SL}(3,\mathbb{R})$ is denoted by $\mathbf{Hyp}_+$. Goldman [7] defined the length parameters ℓ, m on $\mathbf{Hyp}_+$ as $\ell(B) = \log\left(\frac{\nu}{\lambda}\right)$, $m(B) = 3\log(\mu)$ where B is a positive hyperbolic element represented by the diagonal matrix (3.2) with $\lambda\,\mu\,\nu = 1$ and $0 < \lambda < \mu < \nu$.

In this paper we will modify Goldman's length parameters ℓ, m to maintain the consistency with the Fenchel-Nielsen's length parameter ℓ. We modify the length parameters ℓ, m as follows ;

$$\ell(B) = \frac{1}{2}\log\left(\frac{\nu}{\lambda}\right), \quad m(B) = \frac{3}{4}\log(\mu). \tag{5.1}$$

For a hyperbolic manifold M, let Ω be the developing image in $\mathbb{H}^2$ and A be an element of the holonomy group $\Gamma \subset \mathbf{PSL}(2,\mathbb{R})$. Then A is a hyperbolic element. The translation length $\ell(A)$ of A is defined by

$$\ell(A) = \inf_{z\in\Omega} d_P(z, A(z))$$

where d_P is the Poincaré metric on Ω. Then the translation length $\ell(A)$ of A is achieved if and only if z lies on the principal line of A which is the line joining the repelling and attracting fixed point of A. From Beardon's book [3], we get the relation

$$\left|\frac{\mathrm{tr}(A)}{2}\right| = \cosh\left(\frac{\ell(A)}{2}\right). \tag{5.2}$$

Since $\cosh^{-1}(t) = \log(t + \sqrt{t^2-1})$ and $|\mathrm{tr}(A)| = \alpha + \alpha^{-1}$ for $\alpha > 1$, Equation (5.2) becomes

$$\frac{\ell(A)}{2} = \log\left(\frac{\alpha+\alpha^{-1}}{2} + \sqrt{\frac{\alpha^2+2+\alpha^{-2}}{4} - 1}\right) = \log(\alpha).$$

Therefore the Fenchel-Nielsen's length parameter ℓ can be defined as

$$\ell(A) = \log(\alpha^2) \tag{5.3}$$

for a hyperbolic element $A \in \mathbf{PSL}(2,\mathbb{R})$ represented by the diagonal matrix (3.1) with $\alpha > 1$.

Theorem 5.1 (Kim [11]). *The modified Goldman's length parameter ℓ is isometrically the same as the Fenchel-Nielsen's length parameter ℓ.*

Proof. Let $B = \varphi(A)$. Since the length parameter ℓ is invariant under the conjugation, consider the identifications $\lambda = \alpha^{-2}$, $\mu = 1$ and $\nu = \alpha^2$ in (4.18). Then we have

$$\ell(B) = \frac{1}{2}\log(\frac{\nu}{\lambda}) = \frac{1}{2}\log(\frac{\alpha^2}{\alpha^{-2}}) = \frac{1}{2}\log(\alpha^4) = \log(\alpha^2) = \ell(A).$$

Therefore the modified Goldman's length parameter $\ell(B)$ is exactly the same parameter to the Fenchel-Nielsen's length parameter $\ell(A)$. □

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