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BURGHELEA-FRIEDLANDER-KAPPELER'S GLUING FORMULA AND ITS APPLICATIONS TO THE ZETA-DETERMINANTS OF DIRAC LAPLACIANS

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ABSTRACT. For the last two decades the eta-invariant of a Dirac operator on a compact manifold with cylindrical ends has been studied in many ways. In this note we survey some of these results and the BFK-gluing formula for zeta-determinants. As applications of the BFK-gluing formula we give some partial results for the zeta-determinant of a Dirac Laplacian to the analogous questions as those given in the case of eta-invariant.

§1 Introduction

In this note we briefly explain the Burghelea-Friedlander-Kappeler's gluing formula (BFK-gluing formula) for zeta-determinants and its applications to the zeta-determinants of Dirac Laplacians on manifolds with cylindrical ends. Originally the BFK-gluing formula contains a constant which is determined by the asymptotic expansion of the zeta-determinant of a one parameter family of elliptic ΨDO's called the Dirichlet-to-Neumann operators. Under the assumption of the product structure near the hypersurface of a given manifold this constant was computed explicitly by the author ([L2]) and this formula can be applied to a compact oriented manifold with boundary having the product structure near the boundary.

On the other hand, for the last two decades the eta-invariant of a Dirac operator has been studied in various ways and many results have been proved by many authors. Moreover, the zeta-determinant of a Dirac Laplacian and the eta-invariant of a Dirac operator are the modulus and phase, up to the value of the zeta-function for a Dirac Laplacian at zero, of the zeta determinant of a Dirac operator, which plays an important role in mathematical physics (*cf.* [SW]). Hence it's natural to consider the analogous questions for zeta-determinant as the questions given in the case of eta-invariant. Hence for the motivation of our works in the zeta-determinant we start from the eta-invariant of a Dirac operator.

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§2 Eta-invariant on a manifold with cylindrical end

Let M be a $4k$-dimensional compact oriented manifold. The intersection form

$$B : H^{2k}(M;\mathbb{Z}) \otimes H^{2k}(M;\mathbb{Z}) \to \mathbb{Z}$$

is defined as follows. For any $[\alpha], [\beta] \in H^{2k}(M;\mathbb{Z})$

$$B([\alpha],[\beta]) = \langle [\alpha \cup \beta], [z] \rangle,$$

where $[z]$ is the fundamental class in $H_{4k}(M;\mathbb{Z})$. Then B is non-degenerate by the Poincaré duality and the signature of M is defined by

$$Sign(M) = \sharp\{\text{positive eigenvalues of } B\} - \sharp\{\text{negative eigenvalues of } B\}.$$

On the other hand, Hirzebruch signature theorem tells that

$$Sign(M) = \int_M L(M),$$

where $L(M)$ is the Hirzebruch L-polynomial consisting of Pontrjagin classes.

However, in case of manifolds with boundary the situation is more complicated. Suppose that M is a $4k$-dimensional compact oriented manifold with boundary Y. We define a bilinear form

$$B : H^{2k}(M,Y;\mathbb{Z}) \otimes H^{2k}(M,Y;\mathbb{Z}) \to \mathbb{Z}$$

in a similar way. For any $[\alpha], [\beta] \in H^{2k}(M,Y;\mathbb{Z})$

$$B([\alpha],[\beta]) = \langle [\alpha \cup \beta], [z] \rangle,$$

where $[z]$ is the fundamental class in $H_{4k}(M,Y;\mathbb{Z})$. In this case, B may be degenerate and there is some topological obstruction for B to be non-degenerate (*cf.* [GS]). Similarly, we define the signature of M by

$$Sign(M) = \sharp\{\text{positive eigenvalues of } B\} - \sharp\{\text{negative eigenvalues of } B\}.$$

But in this case, $Sign(M) - \int_M L(M) \neq 0$ and the correction term on the boundary Y was given by Atiyah, Patodi, Singer in [APS]. We choose a metric on Y and define a Dirac operator on the space of even forms on Y as follows.

$$D : \Omega^{even}(Y) \to \Omega^{even}(Y)$$

$$D(f_{2p}) = (-1)^{k+p+1}(*d - d*)f_{2p}, \tag{2.1}$$

where $f_{2p} \in \Omega^{2p}(Y)$ and $*$ is the Hodge $*$-operator on Y. Then D is the 1st order elliptic differential operator having discrete spectrum and the eta function is defined by

$$\eta_D(s) = \sum_{0 \neq \lambda_i \in Spec(D)} sign(\lambda_i)|\lambda_i|^{-s}.$$

It is known that $\eta_D(s)$ is regular for $Res > dimY$ and admits the meromorphic continuation to the whole complex plane. It is a non-trivial fact that $\eta_D(s)$ has a regular value at 0. We call $\eta_D(0)$ the eta-invariant associated to D. The famous Atiyah-Patodi-Singer theorem ([APS]) is that

$$Sign(M) = \int_M L(M) - \eta_D(0).$$

For the motivation of a Dirac operator on a manifold with cylindrical ends let us consider the operator D in (2.1).

$$(-1)^{k+p+1}(*d - d*)|_{\Omega^{2p}} = (-1)^{k+p}(d + d^*) * |_{\Omega^{2p}}.$$

For any $\omega \in \Omega^{4k-1-2p}(Y)$,

$$\begin{aligned}
(d + d^*)\omega &= \sum_{n=1}^{4k-1} (e_n \vee \nabla_{e_n}\omega - e_n \angle \nabla_{e_n}\omega) \\
&= \sum_{n=1}^{4k-1} e_n \cdot \nabla_{e_n}\omega \\
&= e_{4k-1} \cdot \nabla_{e_{4k-1}}\omega + \sum_{n=1}^{4k-2} e_n \cdot \nabla_{e_n}\omega \\
&= e_{4k-1} \cdot \left(\nabla_{e_{4k-1}} + \sum_{n=1}^{4k-2} e_n \cdot e_{4k-1} \cdot \nabla_{e_n} \right) \omega, \qquad (2.2)
\end{aligned}$$

where $\{e_1, e_2, \cdots, e_{4k-1}\}$ is a local orthonormal frame for the tangent bundle of Y, ∇ is the Levi-Civita connection on Y, $\angle$ is the interior product and $\cdot$ is the Clifford multiplication on the space of differential forms identified with the Clifford algebra.

Now we are going to discuss the eta-invariant on a manifold with boundary. Suppose that (M, g) is an n-dimensional manifold with boundary Y and $E \to M$ is a Clifford module bundle. Choose a collar neighborhood N of Y which is diffeomorphic to $(-1, 0] \times Y$. We assume that the metric g is a product one on N and the bundle $E \to M$ also has the product structure on N in the following sense. $E|_N = p^*(E|_Y)$, where $p : (-1, 0] \times Y \to Y$ is the natural projection. From now on we consider a Dirac operator D acting on smooth sections of E having the following properties on N.

$$D|_N = G(\partial_u + B), \tag{2.3}$$

where $G : E|_Y \to E|_Y$ is a bundle automorphism, B is a Dirac operator on Y, ∂_u is the outward normal derivative on N and they satisfy the following properties.

$$G^2 = -Id, \quad G^* = -G, \quad GB = -BG. \tag{2.4}$$

Obviously the Dirac operator given in (2.2) satisfies the form (2.3) and (2.4). It can be checked that D is a symmetric operator and D^2 $(= DD^* = D^*D)$ is called the Dirac Laplacian. On the tubular neighborhood N,

$$D^2 = -\partial^2 + B^2.$$

To obtain a discrete spectrum of D we need a proper boundary condition on Y. The most useful one is given by Atiyah, Patodi and Singer. Since B is a 1st order elliptic differential operator on a closed manifold, the spectrum of B is distributed from $-\infty$ to ∞. We denote by

$$P_> \quad (P_<) : C^\infty(Y) \to C^\infty(Y)$$

the orthogonal projection onto the space spanned by the positive (negative) eigensections of B. If $kerB$ is non-trivial, this boundary condition is not enough. If $Bf = 0$, then by (2.4) $BGf = -GBf = 0$ and hence $kerB$ is always an even dimensional vector space. (In fact, G gives the symplectic structure on $kerB$.) We choose a unitary involution $\sigma : kerB \to \ker B$ which anticommutes with G, *i.e.* $\sigma^2 = Id_{kerB}$, $G\sigma = -\sigma G$ and σ is unitary. There are many choices for such σ's. Note that $ker(Id - \sigma)$ is the $(+1)$- eigenspace of σ and $ker(Id + \sigma)$ is the (-1)-eigenspace of σ. We put

$$P_\sigma = P_< + \text{ projection on } ker(Id + \sigma)$$

and define D_{P_σ} by the Dirac operator D with the domain

$$Dom(D_{P_\sigma}) = \{\phi \in C^\infty(M) \mid P_\sigma(\phi|_Y) = 0\}.$$

Then $D_{P_\sigma} : Dom(D_{P_\sigma}) \to C^\infty(M)$ is an essential self-adjoint operator having discrete spectrum. We define the eta-function

$$\eta_{D_{P_\sigma}}(s) = \sum_{0 \neq \lambda_i \in Spec(D_{P_\sigma})} sign(\lambda_i)|\lambda_i|^{-s}$$

and obtain the eta-invariant $\eta_{D_{P_\sigma}}(0)$ associated to D_{P_σ}.

Next, we are going to discuss the adiabatic limit of an eta-invariant. We denote by $M_r := M \cup_Y [0, r] \times Y$ and similarly $M_\infty := M \cup_Y [0, \infty) \times Y$. Then from the product structure on N we can extend the Dirac operator D and the Clifford module bundle E to the Dirac operator D_r and the Clifford module bundle E_r on M_r. Now we consider the bundle $E_r \to M_r$ and the Dirac operator D_{r,P_σ} : $Dom(D_{r,P_\sigma}) \to C^\infty(M_r)$. Then the basic questions are :
(1) How does a spectral invariant change when M is stretched to M_r ?
(2) What is the limit of a spectral invariant as $r \to \infty$?

The followings are some known results concerning these questions.
(1) $\eta_{D_{r,P_\sigma}}(0)$ does not depend on the cylinder length r. (Müller in [M])
(2) If σ_1 and σ_2 are two unitary involutions on $kerB$ satisfying $G\sigma_i = -\sigma_i G$ $(i = 1, 2)$,

$$\eta_{D_{P_{\sigma_1}}}(0) - \eta_{D_{P_{\sigma_2}}}(0) \equiv -\frac{1}{\pi i} \log Det(\sigma_1\sigma_2|_{ker(G-i)}) \quad (mod\mathbb{Z}).$$

(Müller in [M], Lesch and Wojciechowski in [LW])
(3) Gluing formula for eta-invariant : Let M be a closed manifold and Y be a hypersurface so that $M - Y$ has two components. We denote by M_1 and M_2 the components of $M - Y$. Choose a collar neighborhood N of Y which is diffeomorphic to $[-1, 1] \times Y$. We assume the product structure on N and consider a Dirac operator satisfying (2.3) and (2.4) on N. We denote by D_1 and D_2 the restriction of D to M_1 and M_2, respectively. Then

$$\eta_D(0) \equiv \eta_{D_{1,P_{\sigma_1}}}(0) + \eta_{D_{2,P_{\sigma_2}}}(0) + \eta(D_{cyl}, \sigma_1, \sigma_2) \quad (mod\mathbb{Z}),$$

where $D_{cyl} = G(\partial_u + B)$ and $\eta(D_{cyl}, \sigma_1, \sigma_2)$ is the eta-invariant on $[-1, 1] \times Y$ with the boundary conditions $Id - P_{\sigma_1}$ on $\{-1\} \times Y$ and $Id - P_{\sigma_2}$ on $\{1\} \times Y$. (Wojciechowski in [W], see also [BL])
(4) Bunke ([B]), Kirk and Lesch ([KL]) showed the gluing formula for eta-invariant in $\mathbb{R}$ (not in $\mathbb{R}/\mathbb{Z}$). Their formulas contain the concepts of spectral flow and Maslov index.

§3 Zeta-determinant of an elliptic operator and the BFK-gluing formula for zeta-determinants

In this section we are going to explain briefly the zeta-determinant of an elliptic operator and the BFK-gluing formula for zeta-determinants. For the motivation of zeta-determinant we start from the ordinary determinant of a linear operator acting on a finite dimensional vector space.

Suppose that V is an n-dimensional vector space and $T : V \to V$ is a linear operator with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. We define $\zeta_T(s)$ by

$$\zeta_T(s) = \lambda_1^{-s} + \cdots + \lambda_n^{-s} = \sum_{i=1}^{n} \lambda_i^{-s}.$$

Then from simple computation we have

$$\zeta_T'(0) = -\sum_{i=1}^{n} \log \lambda_i = -\log(\lambda_1 \cdots \lambda_n)$$

and

$$e^{-\zeta_T'(0)} = \lambda_1 \cdots \lambda_n = detT. \tag{3.1}$$

The formula (3.1) can be generalized to an operator having infinitely many eigenvlaues.

Let M be a compact closed manifold and $E \to M$ be a vector bundle. Suppose that $A : C^\infty(M) \to C^\infty(M)$ is a positive definite elliptic pseudo-differential operator with eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$. We define

$$\zeta_A(s) := \sum_{i=1}^{\infty} \lambda_i^{-s}.$$

Then it was shown by Seeley ([S]) that $\zeta_A(s)$ is regular for $Res > \frac{dimM}{ord(A)}$ and admits the meromorphic continuation to the whole complex plane having a regular value at $s = 0$. We define the zeta-determinant of A by

$$Det_\zeta A = e^{-\zeta_A'(0)} \text{ or } \log Det_\zeta A = -\zeta_A'(0).$$

The zeta-determinant was introduced by Ray and Singer ([RS]) in defining the analytic torsion, which is the analytic counterpart of the Reidemeister torsion.

Example : Suppose that A is the Laplacian acting on smooth functions on S^1, *i.e.*,

$$A = -\frac{\partial^2}{\partial \theta^2} : C^\infty(S^1) \to C^\infty(S^1).$$

The the spectrum of A is

$$Spec(A) = \{0, 1^2, (-1)^2, 2^2, (-2)^2, \cdots\}$$

and

$$\zeta_A(s) = 1^{2s} + (-1)^{2s} + 2^{2s} + (-2)^{2s} + \cdots = 2 \cdot \zeta_R(2s),$$

where $\zeta_R(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, the Riemann zeta function. It is a well-known fact that $\zeta_R'(0) = -\frac{1}{2}\log 2\pi$. Hence $\zeta_A'(0) = 4 \cdot (-\frac{1}{2}) \cdot \log 2\pi$ and $Det_\zeta A = e^{-\zeta_A'(0)} = 4\pi^2$.

Next, we are going to explain the BFK-gluing formula for zeta-determinants. Let (M, g) be a compact oriented manifold with boundary Z (Z may be empty) and Y be a hypersurface of M such that $Y \cap Z = \emptyset$ and $M - Y$ has two components. Here we don't assume the product structure near Y. We denote by M_1, M_2 the closure of each component, *i.e.* $M = M_1 \cup_Y M_2$. Without loss of generality we asume that $Z \subset M_2$. Let Δ_M be a Laplacian on M and Δ_{M_1}, Δ_{M_2} be the restriction of Δ_M to M_1, M_2, respectively. Choose an elliptic boundary condition P_0 on Z and we assume that Δ_{M,P_0} is an invertible operator. We denote by P_{Dir} the Dirichlet boundary condition on Y and denote by $\Delta_{M_i,P_{Dir}}$ $(i = 1, 2)$ the Laplacian Δ_{M_i} with the domain

$$Dom(\Delta_{M_1}) = \{\phi \in C^\infty(M_1) \mid \phi|_Y = 0\},$$

$$Dom(\Delta_{M_2}) = \{\phi \in C^\infty(M_2) \mid \phi|_Y = 0, P_0(\phi|_Z) = 0\}.$$

Now we define the correction operator $R : C^\infty(Y) \to C^\infty(Y)$ called the Dirichlet-to-Neumann operator as follows. For any $f \in C^\infty(Y)$ choose $\phi_1 \in C^\infty(M_1)$ and $\phi_2 \in C^\infty(M_2)$ satisfying

$$\Delta_{M_1}(\phi_1) = 0, \quad \Delta_{M_2}(\phi_2) = 0, \quad \phi_1|_Y = \phi_2|_Y = f \quad \text{and} \quad P_0(\phi_2|_Z) = 0. \tag{3.2}$$

Both ϕ_1 and ϕ_2 satisfying (3.2) exist uniquely from the Dirichlet boundary condition and the boundary condition P_0. Then we define $Q_1(f)$ and $Q_2(f)$ by

$$Q_1(f) := (\partial_u \phi_1)|_Y, \quad Q_2(f) := (\partial_u \phi_2)|_Y \quad \text{and}$$

$$R(f) := Q_1(f) - Q_2(f) = (\partial_u \phi_1)|_Y - (\partial_u \phi_2)|_Y,$$

where ∂_u is the unit outward normal derivative to M_1. R is known to be a positive definite, invertible, elliptic pseudo-differential operator of order 1 ([L1]) and the following theorem is called the BFK-gluing formula ([BFK], [L1]).

Theorem 3.1. *With the same notation as above we have*

$$\log Det_{\zeta}\Delta_{M,P_0} = c + \log Det_{\zeta}\Delta_{M_1,P_{Dir}} + \log Det_{\zeta}\Delta_{M_2,P_{Dir},P_0} + \log Det_{\zeta} R,$$

where c is determined by the asymptotic symbol of the operator R.

This theorem holds in a general compact oriented manifold (with boundary). If we assume the product structure near Y so that the Laplacian Δ_M has the form $-\partial_u^2 + \Delta_Y$ near Y, it was shown by the author ([L2]) that $c = -\log 2 \cdot (\zeta_{\Delta_Y}(0) + dimker\Delta_Y)$. In the next section we give some results about the zeta-determinant of a Dirac Laplacian as applications of Theorem 3.1.

§4 **Applications of the BFK-gluing formula**

In this section we start from showing the relation between the eta-invariant of a Dirac operator and the zeta-determinant of a Dirac Laplacian (*cf.* [SW]). Suppose that D is a Dirac operator with positive eigenvalues $\{\lambda_k \mid k = 1, 2, 3, \cdots\}$ and negative eigenvalues $\{-\mu_k \mid k = 1, 2, 3, \cdots\}$. Then we have

$$\eta_D(s) = \sum_{k=1}^{\infty} \lambda_k^{-s} - \sum_{k=1}^{\infty} \mu_k^{-s}, \quad \zeta_{D^2}(s) = \sum_{k=1}^{\infty} \lambda_k^{-2s} + \sum_{k=1}^{\infty} \mu_k^{-2s},$$

$$\zeta_D(s) = \sum_{k=1}^{\infty} \lambda_k^{-s} + \sum_{k=1}^{\infty} (-\mu_k)^{-s},$$

where all three functions are holomorphic for $Res > dimM$. Then

$$\begin{aligned} \zeta_D(s) &= \sum_k \left(\frac{\lambda_k^{-s} + \mu_k^{-s}}{2} + \frac{\lambda_k^{-s} - \mu_k^{-s}}{2} \right) + (-1)^{-s} \sum_k \left(\frac{\lambda_k^{-s} + \mu_k^{-s}}{2} - \frac{\lambda_k^{-s} - \mu_k^{-s}}{2} \right) \\ &= \frac{1}{2} \left(\zeta_{D^2}(\frac{s}{2}) + \eta_D(s) \right) + \frac{1}{2} e^{-i\pi s} \left(\zeta_{D^2}(\frac{s}{2}) - \eta_D(s) \right). \end{aligned}$$

Taking derivative with respect to s in $Res > dimM$ we have

$$\begin{aligned} \zeta'_D(s) = \frac{1}{2} \left(\frac{1}{2} \zeta'_{D^2}(\frac{s}{2}) + \eta'_D(s) \right) - \frac{i\pi}{2} e^{-i\pi s} \left(\zeta_{D^2}(\frac{s}{2}) - \eta_D(s) \right) \\ + \frac{1}{2} e^{-i\pi s} \left(\frac{1}{2} \zeta'_{D^2}(\frac{s}{2}) - \eta'_D(s) \right) \end{aligned}$$

After taking meromorphic continuation and putting $s = 0$, we have

$$\zeta'_D(0) = \frac{1}{2} \zeta'_{D^2}(0) - \frac{i\pi}{2} \left(\zeta_{D^2}(0) - \eta_D(0) \right)$$

and hence

$$Det_\zeta D = e^{-\zeta'_D(0)} = e^{\frac{\pi i}{2}(\zeta_{D^2}(0)-\eta_D(0))} \cdot e^{-\frac{1}{2}\zeta'_{D^2}(0)}.$$

This fact implies that the zeta determinant of a Dirac Laplacian and the eta invariant of a Dirac operator are the modulus and phase of the zeta-determinant of a Dirac operator up to $\frac{\pi}{2}\zeta_{D^2}(0)$, which is manipulated more or less easily (*cf.* (Appendix in [PW])). Hence it is natural to consider the analogous questions for the zeta-determinant of a Dirac Laplacian as the questions given in the eta-invariant. More precisely we can ask the similar questions for zeta-determinant as those given in Section 2.

Let us assume the product structure near Y so that $\Delta_M|_N = -\partial_u^2 + \Delta_Y$. The following results, which can be compared with the results given in Section 2, are partial answers of the zeta-determinants of Dirac Laplacians for these questions.
(1) The constant in Theorem 3.1 is $c = -\log 2 \cdot (\zeta_{\Delta_Y}(0) + dimker\Delta_Y)$ and hence

$$\log Det_\zeta \Delta_{M,P_0} = -\log 2 \cdot (\zeta_{\Delta_Y}(0) + dimker\Delta_Y) + \log Det_\zeta \Delta_{M_1,Dir} + \log Det_\zeta \Delta_{M_2,P_{Dir},P_0} + \log Det_\zeta R.$$

(Lee in [L2])
(2) Supose that (M,g) is a compact closed Riemannian manifold and Y be a hypersurface so that $M-Y$ has two components, denoted by M_1 and M_2 as above. Choose a collar neighborhood N diffeomorphic to $[-1,1]\times Y$ and assume the product structure on N. We denote by M_r the compact manifold without boundary obtained by attaching $[-r-1, r+1]\times Y$ on $M - (-\frac{1}{2},\frac{1}{2})\times Y$ by identifying $[-1,-\frac{1}{2}]\times Y$ with $[-r-1,-r-\frac{1}{2}]\times Y$ and $[\frac{1}{2},1]\times Y$ with $[r+\frac{1}{2}, r+1]\times Y$. We also denote by $M_{1,r}$, $M_{2,r}$ the manifolds with boundary which are obtained by attaching $[-r,0]\times Y$, $[0,r]\times Y$ on M_1, M_2 by identifying Y with $\{-r\}\times Y$ and Y with $\{r\}\times Y$, respectively. Then the Dirac operator D and the bundle E can be naturally extended to D_r, E_r on M_r and D_r has, on the cylinder part, the form $D_r = G(\partial_u + B)$. We assume that $kerB = \{0\}$ and there is no extended L^2-solutions of D_∞ on $M_\infty := M\cup_Y [0,\infty)\times Y$ (for definition see [APS] or [BW]). Then we have :

(a) $$\lim_{r\to\infty}\left\{\log Det_\zeta D^2_{M_r} - \log Det_\zeta D^2_{M_{1,r},P_<} - \log Det_\zeta D^2_{M_{2,r},P_>}\right\} = -\log 2\cdot\zeta_{B^2}(0).$$

(Park and Wojciechowski in [PW], Lee in [L3]).

(b) If $dim Y$ is even, as $r \to \infty$

$$\log Det_{\zeta} D^2_{M_{1,r},P_<} \sim \zeta_{B^2}(-\frac{1}{2}) \cdot r +$$

$$\left(-\frac{1}{2}\log 2 \cdot \zeta_{B^2}(0) + \log Det_{\zeta} D^2_{M_{1,P_{Dir}}} - \frac{1}{4}\log Det_{\zeta} B^2 + \log Det_{\zeta}(Q_1 + |B|)\right)$$

$$+ \quad O(\frac{1}{r}).$$

If $dim Y$ is odd, as $r \to \infty$

$$\log Det^2_{\zeta} D_{M_{1,r},P_<} \sim A_1 \cdot r +$$

$$\left(-\frac{1}{2}\log 2 \cdot \zeta_{B^2}(0) + \log Det_{\zeta} D^2_{M_{1,P_{Dir}}} - \frac{1}{4}\log Det_{\zeta} B^2 + \log Det_{\zeta}(Q_1 + |B|)\right)$$

$$+ \quad O(\frac{1}{r}),$$

where $A_1 = \frac{d}{ds}(s \cdot \zeta_{B^2}(s - \frac{1}{2}))|_{s=0} + (s \cdot \zeta_{B^2}(s - \frac{1}{2}))|_{s=0} \cdot (\frac{1}{\sqrt{\pi}}\Gamma'(\frac{1}{2}) + \gamma + 2)$

with γ the Euler constant (Lee in [L4]).

(3) Suppose that (M, g) is a compact oriented manifold with boundary Y and σ_1, σ_2 are two unitary involutions acting on $ker B$ with $G\sigma_i = -\sigma_i G$. Assume that $D_{P_{\sigma_1}}$ and $D_{P_{\sigma_2}}$ are invertible operators. Then

$$\frac{Det_{\zeta} D_{P_{\sigma_1}}}{Det_{\zeta} D_{P_{\sigma_2}}} = det\left(I + 2 \cdot \frac{I + C(0)}{2}(\sigma_2 - \sigma_1)(C(0) - \sigma_2)^{-2}\frac{I + C(0)}{2}\right)|_{ker B}.$$

Here $C(0) : ker B \to ker B$ is the scattering matrix determined by D_∞ on M_∞ (see [M] for details). Moreover, $C(0)$ is a unitary operator with $C(0)^2 = Id_{ker B}$, $GC(0) = -C(0)G$ and hence $C(0)$ is the natural choice of a unitary involution anticommuting with G on $ker B$ (Lee in [L5]).

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