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ON SEMIALGEBRAIC TRANSFORMATION GROUPS II

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ABSTRACT. We survey some recent developments in semialgebraic transformation group theory.

CONTENTS

INTRODUCTION

In "On Semialgebraic Transformation Groups I" [25], the authors discussed very shortly the theory of semialgebraic transformation groups. In this article, as a continuation of [25], we survey some recent results on equivariant semialgebraic homotopy theory, equivariant semialgebraic vector bundles and equivariant Whitehead groups of semialgebraic G-sets. In the final section we treat some problems in Nash transformation group theory.

Recall that a semialgebraic G-set is called **affine** if it is semialgebraically G-homeomorphic to a G-invariant semialgebraic set in some semialgebraic representation space of G. We remember that every semialgebraic G-set is affine when G is a compact semialgebraic linear group, see [25, Theorem 4.3] or [24].

1. SEMIALGEBRAIC G-HOMOTOPY THEORY

In this section we discuss semialgebraic G-homotopies of semialgebraic G-maps.

The definition in semialgebraic homotopy theory is similar to that of topological homotopy theory, except that topological spaces are replaced by semialgebraic sets and continuous maps by semialgebraic maps. For example, let M and N be semialgebraic G-sets. Two semialgebraic G-maps $f, g\colon M \to N$ are said to be **semialgebraically G-homotopic** if there is a semialgebraic G-map $H\colon M \times I \to N$ such that $H_0 = f$, $H_1 = g$ with trivial G-action on I. The map H is called a **semialgebraic G-homotopy** from f to g. Note that the interval $I = [0, 1]$ is semialgebraic.

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A semialgebraic G-subset A of M is called a **semialgebraic strong G-deformation retract** of M if there exists a semialgebraic G-homotopy $H\colon M \times I \to M$ such that $H_0 = \mathrm{id}_M$ and H_1 is a retraction from M to A and $H(a,t) = a$ for all $(a,t) \in A \times I$.

Let A be a G-subset of a finite G-CW complex X. The G-**star** of A in X denoted by $\mathrm{St}_X(A)$ is the union of all open G-cells c of X such that $\overline{c} \cap A \neq \emptyset$. Clearly $\mathrm{St}_X(A)$ is an open G-neighborhood of A.

Let M be an affine semialgebraic G-set and A a closed semialgebraic G-subset of M. We can give a finite open G-CW complex structure $(X,\ \{c_i \mid i = 1, \ldots, n\})$ of M compatible with A by Theorem 4.2 of [25]. Then A is a semialgebraic strong G-deformation retract of $\mathrm{St}_{X'}(A)$ where X' is a barycentric subdivision of X. Thus we have the following.

Proposition 1.1 ([22]). *Let G be a compact semialgebraic group. Let A be a closed semialgebraic G-subset of a semialgebraic G-set M. Then there exists a G-invariant semialgebraic open neighborhood $V = \mathrm{St}_{X'}(A)$ of A in M such that A is a semialgebraic strong G-deformation retract of V.*

We know the following: a pair (M, A) of topological spaces has the homotopy extension property if and only if $(A \times I) \cup (M \times \{0\})$ is a retract of $M \times I$. In this case A is closed. Moreover any CW complex pair (X, A) has the homotopy extension property, see [8]. We have the similar result in the equivariant semialgebraic category as follows.

Theorem 1.2 ([22]). *Let G be a compact semialgebraic group. If M is a semialgebraic G-set and A is a closed semialgebraic G-subset of M, then $(A \times I) \cup (M \times \{0\})$ is a semialgebraic strong G-deformation retract of $M \times I$. In particular, (M, A) has the semialgebraic G-homotopy extension property.*

The above theorem is proved from Proposition 1.1 and the equivariant semialgebraic Urysohn's lemma of semialgebraic G-sets ([25, Proposition 3.2]).

Let M be a semialgebraic G-set. We define the **core** of a finite open G-CW complex $M \subset \Omega$, denoted by $\mathrm{co}(M)$, to be the G-CW subcomplex of M consisting of all open G-cells $c = G\sigma$ of M which has a compact closure in M. Then $\mathrm{co}(M)$ is the unique maximal compact G-CW subcomplex of M. If M is a semialgebraic G-set with the G-CW complex structure as in Theorem 4.2 of [25], then we have $\mathrm{co}(M) = \pi^{-1}(\mathrm{co}(M/G))$ where $\pi\colon M \to M/G$ is the orbit map.

Since the star of $(\mathrm{co}(M))$ is M, there exists a semialgebraic strong G-deformation retraction $H\colon M \times I \to M$ from M to $\mathrm{co}(M)$ by Proposition 1.1. Moreover, if A is a closed G-CW subcomplex of M then the restriction $r_M|_A$ is a semialgebraic G-retraction from A to $\mathrm{co}(A)$. Thus we have the following proposition.

Proposition 1.3 ([22]). *Let M be a semialgebraic G-set. Then there exist a compact semialgebraic G-subset C of M and a semialgebraic strong G-deformation retract $R\colon M \times I \to M$ such that $R_0 = R(\cdot, 0) = \mathrm{id}_M$, $R(c,t) = c$ for all $(c,t) \in C \times I$ and $R_1 = R(\cdot, 1) = r\colon M \to C$ is a semialgebraic G-retraction.*

Moreover the inclusion $i\colon C \hookrightarrow M$ is a semialgebraic G-homotopy inverse of r. In particular, r is a semialgebraic G-homotopy equivalence.

Now we consider the set of semialgebraic G-homotopy classes of semialgebraic G-maps between two semialgebraic G-sets which are not necessarily compact. Let (M, A) and (N, B) be two pairs of semialgebraic G-sets. Let C be a semialgebraic G-subset of

M and let us fix a semialgebraic G-map $h\colon C \to N$ such that $h(C \cap A) \subset B$. From now on we consider G-maps from (M, A) to (N, B) which extend h.

We call any two such semialgebraic G-extensions $f, g\colon (M, A) \to (N, B)$ of h are **semialgebraically G-homotopic relative to** C if there exists a semialgebraic G-homotopy $H\colon (M \times I, A \times I) \to (N, B)$ such that $H_0 = f$, $H_1 = g$ and $H(c, t) = h(c)$ for $(c, t) \in C \times I$.

Let $[(M, A), (N, B)]^{G,h}_{sem}$ (resp. $[(M, A), (N, B)]^{G,h}_{top}$) denote the set of relative semialgebraic (resp. topological) G-homotopy classes of semialgebraic (resp. continuous) G-maps from (M, A) to (N, B) which extend h. We have a canonical map

$$\mu\colon [(M, A), (N, B)]^{G,h}_{sem} \to [(M, A), (N, B)]^{G,h}_{top}$$

which sends the semialgebraic G-homotopy class $[f]_{sem}$ of a semialgebraic G-map f to the topological G-homotopy class $[f]_{top}$ of f. Then we have the following **comparison theorem for G-maps**.

Theorem 1.4 ([22])**.** *Let G be a compact semialgebraic group. Let (M, A) and (N, B) be pairs of affine semialgebraic G-sets. If A and C are closed semialgebraic G-subsets of M and $h\colon (C, C \cap A) \to (N, B)$ is a semialgebraic G-map then the canonical map μ is bijective.*

It is proved by the equivariant semialgebraic homotopy extension property (Theorem 1.2) and the equivariant semialgebraic Uryshon's lemma ([25, Proposition 3.2]). Theorem 1.4 implies that any topological G-homotopy class of a continuous G-map between two affine semialgebraic G-sets can be represented by a semialgebraic G-map. As a corollary of Theorem 1.4, we have the semialgebraic version of the **Tietze-Gleason extension theorem** ([2, I.2.3]) as follows.

Corollary 1.5 ([24])**.** *Let G be a compact semialgebraic group. Let M be an affine semialgebraic G-set. Let Ω be a semialgebraic G-representation space with underlying space $\mathbb{R}^n$. Let C be a closed semialgebraic G-subset of M and $h\colon C \to \Omega$ a semialgebraic G-map. Then there exists a semialgebraic G-extension $f\colon M \to \Omega$ of h.*

2. SEMIALGEBRAIC G-VECTOR BUNDLES

In this section we discuss semialgebraic G-vector bundles over semialgebraic G-sets. The definition of a semialgebraic vector bundle is similar to that of a topological vector bundle, see [1]. We remark that the total space of a semialgebraic vector bundle, in the definition of it (see, [1, p.332]), is not a semialgebraic set but a semialgebraic space. But, since every (nonequivariant) semialgebraic vector bundle has a semialgebraic classifying map (see, [1, Corollary 12.7.5]), we can view the total space is also a semialgebraic set.

Let G be a semialgebraic group. A G-vector bundle $\xi = (E, p, M)$ is a **semialgebraic G-vector bundle** if

(a) E and M are semialgebraic G-sets.
(b) $p\colon E \to M$ is a semialgebraic G-map, and g sends $E|_x$ to $E|_{gx}$ linearly for all $g \in G$.
(c) (E, p, M) is a semialgebraic vector bundle when we forget the action.

If ξ and ξ' are two semialgebraic G-vector bundles over an affine semialgebraic G-set M, then $\xi \oplus \xi'$, $\xi \otimes \xi'$, $\mathrm{Hom}(\xi, \xi')$ and the dual bundle $\xi^\vee$ are semialgebraic G-vector bundles over M. If $f\colon M \to N$ is a semialgebraic G-map between two semialgebraic

G-sets and ξ is a semialgebraic G-vector bundle over N, the pull-back $f^*(\xi)$ is also a semialgebraic G-vector bundle over M.

Let Ω be an orthogonal semialgebraic representation space of a semialgebraic group G with underlying vector space $\mathbb{R}^n$. Then the grassmanian manifold $\mathbb{G}(\Omega, k)$ is a nonsingular algebraic G-variety and thus a Nash G-manifold. In particular, $\mathbb{G}(\Omega, k)$ is an affine semialgebraic G-set. Moreover the universal bundle $\gamma(\Omega, k)$ over $\mathbb{G}(\Omega, k)$ is a strongly algebraic G-vector bundle and thus a strongly semialgebraic G-vector bundle.

Theorem 2.1 ([3]). *Let ξ be a semialgebraic G-vector bundle over an affine semialgebraic G-set M. Then there exists a semialgebraic G-map $f\colon M \to \mathbb{G}(\Omega, k)$ such that ξ is semialgebraically G-isomorphic to $f^*(\gamma(\Omega, k))$ for some Ω and k.*

Theorem 2.1 says that every semialgebraic G-vector bundle over an affine semialgebraic G-set is strongly semialgebraic.

Now we compare the set of semialgebraic G-isomorphism classes of semialgebraic G-vector bundles with that of topological isomorphism classes of topological G-vector bundles over a semialgebraic G-set.

Let G be a compact semialgebraic group and let M be a semialgebraic G-set. The set $\mathrm{Vect}_G^{sem}(M)$ (resp. $\mathrm{Vect}_G^{top}(M)$) denotes the set of semialgebraic (resp. topological) G-isomorphism classes of semialgebraic (resp. topological) G-vector bundles over M. We have the canonical map

$$\kappa\colon \mathrm{Vect}_G^{sem}(M) \to \mathrm{Vect}_G^{top}(M)$$

which sends the semialgebraic G-isomorphism classes $[\xi]_{sem}$ of a semialgebraic G-vector bundle ξ over M to the topological G-isomorphism class $[\xi]_{top}$ of ξ.

Then we have the following **comparison theorem for G-vector bundles**.

Theorem 2.2 ([3]). *Let G be a compact semialgebraic group and let M be an affine semialgebraic G-set. Then the canonical map κ is bijective.*

As an application of the comparison theorem for G-vector bundles, we have the following corollary.

Corollary 2.3 ([3]). *Let ξ and η be semialgebraic G-vector bundles over an affine semialgebraic G-set. If $\xi \cong_G^{top} \eta$ then $\xi \cong_G^{sem} \eta$.*

As another application of Theorem 2.2, we have the equivariant semialgebraic version of the covering homotopy property for semialgebraic G-vector bundles as follows.

Corollary 2.4 ([3]). *Let $f, h\colon M \to N$ be semialgebraic G-maps between semialgebraic G-sets. Let ξ be a semialgebraic G-vector bundle over N. If M is affine and f is G-homotopic to h, then the pull-back bundles $f^*(\xi)$ and $h^*(\xi)$ are semialgebraically G-isomorphic.*

3. Equivariant Whitehead groups of semialgebraic G-sets

The notion of simple homotopy and Whitehead torsion have been generalized to the equivariant case in topological category, see [9].

In this section we consider the equivariant generalizations of them to the semialgebraic category. Namely, we define the equivariant Whitehead group of a semialgebraic G-set and the Whitehead torsion of a G-homotopy equivalence between semialgebraic G-sets. Moreover, we prove the semialgebraic invariance of the equivariant Whitehead torsion.

The basic ingredients for the development are the existence of equivariant semialgebraic G-CW complex structure of a semialgebraic G-set [25, 21, 24] and equivariant semialgebraic homotopy theory introduced in Section 1. By Theorem 4.2 of [25], any semialgebraic G-set M has a (finite open) semialgebraic G-CW complex structure X. We remark that the (equivariant) Whitehead group is defined on a compact (G-) CW complex [9, 17]. However, in general, a semialgebraic G-set has a finite open G-CW complex structure which may not be compact.

Recall that the **core** of X, denoted by $\mathrm{co}(X)$, is a maximal compact G-CW subcomplex of X. It is shown, in Section 1, that there exists a semialgebraic G-retract $r_X\colon X \to \mathrm{co}(X)$ such that the inclusion map $i_X\colon \mathrm{co}(X) \hookrightarrow X$ is the semialgebraic G-homotopy inverse of r_X. Since $\mathrm{co}(X)$ is a compact G-CW complex, the equivariant Whitehead group $\mathrm{Wh}_G(\mathrm{co}(X))$ can be defined as in [9]. We define the equivariant Whitehead group of a semialgebraic G-set M to be

$$\mathrm{Wh}_G(M) := \mathrm{Wh}_G(\mathrm{co}(X_0))$$

where X_0 is a preferred semialgebraic G-CW complex structure on M. Let X be another semialgebraic open G-CW complex structure on M. For simplicity let us assume that both X and X_0 have the same underlying topological space M. Let $\lambda^X_{X_0}$ denote the composition

$$\mathrm{co}(X) \overset{i_X}{\hookrightarrow} X = X_0 \overset{r_{X_0}}{\to} \mathrm{co}(X_0).$$

Then $\lambda^X_{X_0}$ induces an isomorphism

$$(\lambda^X_{X_0})_*\colon \mathrm{Wh}_G(\mathrm{co}(X)) \to \mathrm{Wh}_G(\mathrm{co}(X_0)),$$

which shows that the definition of $\mathrm{Wh}_G(M)$ is independent of the choice of a semialgebraic open G-CW complex structure on M.

For a G-homotopy equivalence $f\colon M \to N$ between two semialgebraic G-sets we define the Whitehead torsion $\tau_G(f)$ of f to be an element in $\mathrm{Wh}_G(M)$ as follows: Choose any semialgebraic open G-CW complex structures X and Y on M and N, respectively. Put $\tilde{f} = r_Y \circ f \circ i_X\colon \mathrm{co}(X) \to \mathrm{co}(Y)$. Then $\tau_G(\tilde{f}) \in \mathrm{Wh}_G(\mathrm{co}(X))$. We define $\tau_G(f)$ by

$$\tau_G(f) = (\lambda^X_{X_0})_*(\tau_G(\tilde{f})) \in \mathrm{Wh}_G(M).$$

Then such defined Whitehead torsion is well-defined, i.e., independent of the choice of semialgebraic open G-CW complex structures, and is a G-equivariant topological property. Namely we have the following theorem.

Theorem 3.1 ([23]). *Let M and N be affine semialgebraic G-sets. For a G-homotopy equivalence $f\colon M \to N$ there is a well-defined Whitehead torsion $\tau_G(f) \in \mathrm{Wh}_G(M)$ and if f is a semialgebraic G-homeomorphism then $\tau_G(f) = 0$.*

Notice that the topological invariance of the Whitehead torsion does not hold in the equivariant case, see Examples I.4.25 and I.4.26 in [17]. Recently, S. Illman proved a similar result in [12] for smooth Lie group (not necessarily compact) actions on proper smooth G-manifolds, which are not necessarily compact but the orbit spaces are compact. That the orbit spaces are compact is used essentially in [12]. On the other hand in Theorem 3.1 (and Theorem 3.2) the acting groups are compact but the semialgebraic G-sets and their orbit spaces are not necessarily compact.

In this section we also discuss the restriction homomorphism in the semialgebraic category with the preferred Whitehead group. We first discuss the operation of restricting the compact (topological) group G to a closed subgroup H of G. Let X be a compact G-CW complex and consider the induced action of H on X. Note that the H-space X does not in general inherit an induced H-CW complex structure, at least not in any natural way, see [10] for an example. However, given a compact G-CW complex X and a closed subgroup H of a compact Lie group G, one can always construct an H-CW complex $\mathrm{R}_H X$ with the same H-homotopy type as the H-space X. Moreover, this can be done in such a way that the functor taking X to $\mathrm{R}_H X$ preserves the topological dimension and the H-orbit types occurring in $\mathrm{R}_H X$ are exactly the same ones as in X, see [10]. We call $\mathrm{R}_H X$ a **preferred H-reduction of** X.

In the case when X is a semialgebraic G-CW complex structure on a semialgebraic G-set M, one can always construct a semialgebraic H-CW complex structure $\mathrm{I}_H X$ of H-space X such that each G-equivariant cell of X is an H-subcomplex of $\mathrm{I}_H X$. We call $\mathrm{I}_H X$ an **identity H-reduction of** X. Let Y denote the union of (open) G-cells of $\mathrm{I}_H X$ which are contained in $|\operatorname{co}(X)|$. Then Y is a compact H-CW subcomplex of $\mathrm{I}_H X$ with the underlying space $|\operatorname{co}(X)|$, and hence Y is a semialgebraic H-CW structure on $|\operatorname{co}(X)| = |\operatorname{co}(X) \cap \mathrm{I}_H X|$. Moreover $\operatorname{co}(X)$, $\mathrm{R}_H(\operatorname{co}(X))$ and $\operatorname{co}(\mathrm{I}_H X)$ have the same H-homotopy type. In particular Y is a preferred H-reduction of G-CW complex $\operatorname{co}(X)$, and thus we denote Y by $\mathrm{R}_H(\operatorname{co}(X))$. Hence, because $\operatorname{co}(X)$ is compact, there is the restriction homomorphism $\mathrm{Res}_H^G\colon \mathrm{Wh}_G(\operatorname{co}(X)) \to \mathrm{Wh}_H(\mathrm{R}_H(\operatorname{co}(X)))$ by [10, 11]. We are now able to define the restriction homomorphism

$$\mathcal{R}es_H^G\colon \mathrm{Wh}_G(\operatorname{co}(X)) \overset{\mathrm{Res}_H^G}{\to} \mathrm{Wh}_H(\mathrm{R}_H(\operatorname{co}(X))) \overset{(i_X)_*}{\to} \mathrm{Wh}_H(\operatorname{co}(I_H X))$$

by $\mathcal{R}es_H^G = (i_X)_* \circ \mathrm{Res}_H^G$, where $i_X\colon \mathrm{R}_H(\operatorname{co}(X)) = \operatorname{co}(X) \hookrightarrow \operatorname{co}(I_H X)$ is the inclusion map. By using the properties of Res_H^G with the fact that we define the Whitehead group $\mathrm{Wh}_G(M)$ of a semialgebraic G-set by $\mathrm{Wh}_G(\operatorname{co}(X))$ for arbitrarily semialgebraic G-CW complex structure on M, we prove the following.

Theorem 3.2 ([23])**.** *Let G be a compact semialgebraic group, and $K < H < G$ be closed semialgebraic subgroups of G. Let M be a semialgebraic G-set, then there exists a well-defined restriction homomorphism*

$$\mathcal{R}es_H^G\colon \mathrm{Wh}_G(M) \to \mathrm{Wh}_H(M).$$

Moreover we have that if $f\colon M \to N$ is a G-homotopy equivalence between semialgebraic G-sets, and $f_H\colon M \to N$ denotes the induced H-homotopy equivalence, then

$$\tau_H(f_H) = \mathcal{R}es_H^G(\tau_G(f)) \in \mathrm{Wh}_H(M).$$

Furthermore, we have $\mathcal{R}es_K^G = \mathcal{R}es_K^H \circ \mathcal{R}es_H^G$.

Remark that $\mathcal{R}es_H^G = \mathrm{Res}_H^G$ when M is compact.

4. Problems on Nash G-manifolds

In this section we treat some problems in Nash transformation group theory. The Nash category lies between the nonsingular algebraic category and the smooth category (in fact, analytic category).

Recall that a smooth submanifold of $\mathbb{R}^n$ is called a **Nash manifold** in $\mathbb{R}^n$ if it is a semialgebraic set in $\mathbb{R}^n$. Some times many people call a Nash manifold in $\mathbb{R}^n$ an affine Nash manifold. So, in this section, all Nash manifolds are affine. A map between two Nash manifolds is called **Nash** if it is a semialgebraic map of the class $\mathcal{C}^\infty$. We remark that a Nash manifold and a Nash map are automatically of the class $\mathcal{C}^\omega$ (analytic). The equivariant Nash terminologies are defined similarly to the equivariant semialgebraic case. Throughout this section G denotes a compact Nash group.

As an equivariant smooth embedding theorem we have the following equivariant Nash embedding problem.

Problem 4.1. Can every Nash G-manifold be Nash G-embedded into some finite dimensional Nash representation space of G?

We give some of the results related to the equivariant embedding problem. Equivariant embedding problems of the G-space in the given category have been studied by many people as follows.

Category	Reference
Smooth	Mostow [19] and Palais [20]
Analytic	Matumoto and Shiota [18], Kankaanrita [14]
Algebraic	[16], [26]
Semialgebraic	Park and Suh [24]
Nash	?

There are some results about Problem 4.1 in some case as follows; a Nash G-manifold M is Nash G-diffeomorphic to a G-invariant Nash submanifold of some Nash representation space of G if M is compact or compactifiable as a Nash G-manifold.

Thus Problem 4.1 is reduced to the following equivariant Nash compactifiable problem.

Problem 4.2. Is every Nash G-manifold compactifiable as a Nash G-manifold?

In nonequivariant case, for each noncompact Nash manifold M, there exists a compact Nash manifold with boundary whose interior is Nash diffeomorphic to M [27].

Another question is whether a given smooth situation be Nash realized.

Problem 4.3. When a given smooth G-manifold is G-diffeomorphic to a Nash G-manifold?

There are some results about this problem in the equivariant algebraic case. Recall that every algebraic G-variety is a Nash G-manifold.

Proposition 4.4. *A closed G-manifold is G-diffeomorphic to a nonsingular algebraic G-variety if one of the following assumption holds.*

(1) $G = 1$ (*see*, [13]).
(2) *G is the product of a group of odd order and 2-torus* ([5], [7]).
(3) *G is finite abelian with cyclic 2-Sylow subgroup or G is S^1.*
(4) *The action of G on the manifold is semifree* ([5]).
(5) *The manifold is of dimension* 2 ([4], [15]).
(6) *The manifold is G-cobordant to a nonsingular algebraic variety* ([6]).

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