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SCALAR CURVATURE AND MINIMAL VOLUMES

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ABSTRACT. On an n-dimensional Riemannian manifold X, we consider the convex combination $\lambda K+(1-\lambda)\frac{s}{n(n-1)}$ of the sectional curvature K and the scalar curvature s, where $\lambda\in[0,1)$. It is shown that if X admits a metric with $\lambda K+(1-\lambda)\frac{s}{n(n-1)}>0$, then so does any manifold obtained from X by surgeries of codimension ≥ 3. This implies the existence of such metrics on certain compact simply connected manifolds of dimension ≥ 5 by using the cobordism argument. Using this we compute the corresponding minimal volumes. As a corollary, we derive that every compact simply connected manifold of dimension ≥ 5 and every compact complex surface of Kodaira dimension ≤ 1 which is not of Class VII collapse with $\lambda K+(1-\lambda)\frac{s}{n(n-1)}$ bounded below.

1. INTRODUCTION

It is a well-known fact that every smooth manifold of dimension greater than 2 admits a metric of negative scalar curvature. But for the existence of positive scalar curvature there are topological obstructions. If a compact oriented 3-manifold admits a metric of positive scalar curvature, then it has no $K(\pi,1)$ component in the unique prime decomposition. On a compact oriented 4-manifold, a necessary condition for the existence of a metric of positive scalar curvature is that Seiberg-Witten invariant should vanish for any spin^c structure and so in particular $\hat{A}$-genus is zero in case of spin manifolds. In higher dimensions, a compact simply connected manifold of dimension $n\geq 5$ admits a metric of positive scalar curvature metric if and only if it is non-spin or α-genus is zero in the spin case, where α-genus is a surjective homomorphism from the spin cobordism ring $\Omega_*^{\mathrm{S}pin}$ onto the K-ring $KO_*(\mathrm{pt})$, generalizing $\hat{A}$-genus.

In an attempt to measure how much the negative scalar curvature is inevitable on a compact manifold X, we consider the Yamabe minimal volume invariant of X defined as,

$$\mathrm{Vol}_s(X) := \inf_g \left\{ \mathrm{Vol}(X,g) \;\middle|\; \frac{s_g}{n(n-1)} \geq -1 \right\},$$

where n is the dimension of the manifold. Note that if X admits a metric with nonnegative scalar curvature, then $\mathrm{Vol}_s(X)=0$. Or more generally we can also consider the λ-mixed minimal volume invariant,

$$\mathrm{Vol}_{\lambda,K,s}(X) := \inf_g \left\{ \mathrm{Vol}(X,g) \;\middle|\; \lambda K_g+(1-\lambda)\frac{s_g}{n(n-1)} \geq -1 \right\},$$

where $0\leq\lambda<1$ is a constant.

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In general, the computation of these invariants is not easy but strikingly Petean [15] proved the following vanishing theorem in high dimensions.

Theorem 1.1 (Petean)**.** *Let X be a simply connected manifold of dimension ≥ 5. Then $Vol_s(X) = 0$.*

In dimension 4, LeBrun [10, 11, 12] obtained the following nontrivial result via Seiberg-Witten theory.

Theorem 1.2 (LeBrun)**.** *Let X be a compact Kähler surface.*

If the Kodaira dimension of X is less than 2, then $Vol_s(X) = 0$.

If the Kodaira dimension of X is equal to 2, then $Vol_s(X) = \frac{2\pi^2}{9}c_1^2(M)$ where M is the minimal model of X, and $Vol_{\frac{1}{2},K,s}(X) \geq \frac{9}{4}Vol_s(X)$. Moreover the equality holds if X is a compact complex-hyperbolic 4-manifold $\mathbb{C}\mathcal{H}_2/\Gamma$.

Following them, we try to compute the λ-mixed minimal volumes for $\lambda \in [0, 1)$.

2. Main Results

We start with the following generalization of Gromov-Lawson surgery theorem [6].

Theorem 2.1. *Let X be a Riemannian manifold with $\lambda K + (1-\lambda)\frac{s}{n(n-1)} > c$ where c is a constant. Then any manifold obtained from X by performing surgeries in codimension ≥ 3 also carries a metric with $\lambda K + (1-\lambda)\frac{s}{n(n-1)} > c$.*

The same is true of the "connected sum" along embedded spheres with trivial normal bundles in codimension ≥ 3.

By a surgery in an embedded k-sphere S^k with a trivial normal bundle, we mean to take out a tubular neighborhood which is diffeomorphic to $S^k \times D^{n-k}$ and glue $D^{k+1} \times S^{n-k-1}$ along the boundary in the obvious way.

The proof is almost same as that of Gromov and Lawson. The key fact here is that $\lambda K + (1-\lambda)\frac{s}{n(n-1)}$ behaves in a similar way as s in the region where the surgery is performed as long as $\lambda \in [0, 1)$.

Then the following corollary is also immediate in view of Gromov-Lawson [6].

Corollary 2.2. *Any compact simply connected non-spin manifold X of dimension ≥ 5 carries a metric with $\lambda K + (1-\lambda)\frac{s}{n(n-1)} > 0$.*

Corollary 2.3. *Any compact simply connected spin manifold X of dimension ≥ 5 with $\alpha(X) = 0$ carries a metric with $\lambda K + (1-\lambda)\frac{s}{n(n-1)} > 0$.*

In particular, every compact simply connected manifold of dimension $5, 6$, or 7 admits a metric with $\lambda K + (1-\lambda)\frac{s}{n(n-1)} > 0$, because the spin cobordism group in dimension $5, 6$, or 7 is trivial.

Now let's consider the minimal volume. Since the volume of the region where the surgery is performed can be taken arbitrarily small, we get

Theorem 2.4. *Let X_1 and X_2 be compact n-manifolds for $n \geq 3$. Then*

$$Vol_{\lambda,K,s}(X_1 \# X_2) \leq Vol_{\lambda,K,s}(X_1) + Vol_{\lambda,K,s}(X_1).$$

The same is true of the "connected sum" along embedded spheres with trivial normal bundles in codimension ≥ 3.

Since $D^{k+1}\times S^{n-k-1}$ for $n-k\geq 3$ admits a metric of positive sectional curvature, we immediately obtain

Corollary 2.5. *Let X' be a compact manifold obtained from X by surgeries in codimension ≥ 3. Then*

$$Vol_{\lambda,K,s}(X')\leq Vol_{\lambda,K,s}(X).$$

By performing "inverse surgeries" on X' we have

Theorem 2.6. *Let X' be a n-dimensional manifold obtained from a compact manifold X by surgeries on spheres of dimension $\neq 1, n-1, n-2$ for $n\geq 4$. Then*

$$Vol_{\lambda,K,s}(X')= Vol_{\lambda,K,s}(X).$$

By using the above results and employing similar ways as Petean [15] and LeBrun [11], we can get the following generalizations.

Theorem 2.7. *Let X be a compact simply connected manifold of dimension ≥ 5. Then $Vol_{\lambda,K,s}(X)=0$.*

Theorem 2.8. *Let X be a compact complex surface of Kodaira dimension ≤ 1, which is not of Class VII. Then $Vol_{\lambda,K,s}(X)=0$.*

A surface of Class VII is by definition a complex surface with Kodaira dimension $-\infty$ and the first betti number equal to 1. The complete classification of these surfaces is still lacking. One of well-known examples is a Hopf surface. It is diffeomorphic to $S^1\times S^3$ and hence has $\mathrm{Vol}_{\lambda,K,s}=0$.

Noting that the blowing-up is the same as taking connected sum with $\overline{\mathbb{C}P^n}$ which has zero minimal volume, from the theorem 2.4 it follows that

Corollary 2.9. *Suppose X is birational to a compact complex-hyperbolic 4-manifold. Then*

$$Vol_{\frac{1}{2},K,s}(X)=\frac{9}{4}\,Vol_s(X)=\frac{\pi^2}{2}c_1^2(M)$$

where M is the minimal model of X.

Finally we remark that these minimal volumes crucially depend on the smooth structure of a manifold. For example, let X be a simply connected complex surface of general type and Y be a simply connected complex surface of Kodaira dimension ≤ 1 with the same geometric genus as X.(Such Y always exists.) Let k be any positive integer such that $c_1^2(X)-c_1^2(Y)+k>0$. Then the 4-manifolds $X_k\equiv X\#(c_1^2(X)-c_1^2(Y)+k)\overline{\mathbb{C}P^2}$ and $Y_k\equiv Y\#k\overline{\mathbb{C}P^2}$ are homeomorphic according to Freedman's classification [4]. But their minimal volumes are different. Indeed

$$\mathrm{Vol}_{\frac{1}{2},K,s}(X_k)\geq\frac{\pi^2}{2}c_1^2(M)>0,$$

where M is the minimal model of X and $\mathrm{Vol}_{\lambda,K,s}(Y_k)=0$ by theorem 2.8. The following theorem also hints that the mixed minimal volume invariant is significant as a smooth invariant rather than a purely topological invariant.

Theorem 2.10. *Assume the 11/8-conjecture is true. Then every smooth compact simply connected 4-manifold is homeomorphic to one which has $Vol_{\lambda,K,s}=0$*

Proof. As suggested in [14], we have to rely on Freedman's [4] and Donaldson's [3] well-known results on the classification of smooth compact simply connected 4-manifolds. Connected sums of $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$ realize all odd intersection forms and all definite intersection forms. The 11/8-conjecture implies every even indefinite intersection form can be realized by taking connected sums of K3 surfaces and $S^2 \times S^2$. All these building blocks have been proved to have $\mathrm{Vol}_{\lambda,K,s} = 0$, hence the theorem follows. □

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