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MAXIMAL HOLONOMY OF INFRA-NILMANIFOLDS WITH $\mathbf{H}_7(\mathbb{H})$-GEOMETRY

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ABSTRACT. This is a brief summary of the recent work on quaternionic Heisenberg manifolds [2]. We introduce the quaternionic Heisenberg group $\mathbf{H}_{4n+3}(\mathbb{H})$ of real dimension $4n+3$ and prove that the maximal order of the holonomy groups of all infra-nilmanifolds with $\mathbf{H}_7(\mathbb{H})$-geometry is 48.

1. INTRODUCTION

The quaternionic Heisenberg group $\mathbf{H}_{4n+3}(\mathbb{H})$ is

$$\mathbf{H}_{4n+3}(\mathbb{H}) = \mathbb{R}^3 \tilde{\times} \mathbb{H}^n$$

with group operation given by

$$(s, \mathbf{q}) \cdot (t, \mathbf{q}') = (s + t + 2\mathrm{Im}\{\mathbf{q}\overline{\mathbf{q}}'\}, \mathbf{q} + \mathbf{q}'),$$

for $\mathbf{q} = (q_1, q_2, \cdots, q_n)$, $\mathbf{q}' = (q_1', q_2', \cdots, q_n') \in \mathbb{H}^n$, where $\mathrm{Im}\{\mathbf{q}\overline{\mathbf{q}}'\}$ is the imaginary part of the quaternion number $q_1\overline{q_1}' + q_2\overline{q_2}' + \cdots + q_n\overline{q_n}'$ seen as an element of $\mathbb{R}^3$. Then $\mathbf{H}_{4n+3}(\mathbb{H})$ is a simply connected 2-step nilpotent Lie group with the center $\mathcal{Z}(\mathbf{H}_{4n+3}(\mathbb{H})) = \mathbb{R}^3$.

Let M be an infra-nilmanifold with $\mathbf{H}_{4n+3}(\mathbb{H})$-geometry; that is, $M = \Pi\backslash\mathbf{H}_{4n+3}(\mathbb{H})$, where $\Pi \subset \mathbf{H}_{4n+3}(\mathbb{H}) \rtimes C$ is a torsion free, discrete subgroup with compact quotient, where C is a compact subgroup of $\mathrm{Aut}(\mathbf{H}_{4n+3}(\mathbb{H}))$. Such a group Π is called an *almost Bieberbach group* (=AB-group).

Let I_{n+1} denote the maximal order of the holonomy groups of all infra-nilmanifolds with $\mathbf{H}_{4n+3}(\mathbb{H})$-geometry. Then the main result of this paper is the following.

Main Theorem. $I_2 = 48$.

According to a recent work of Kim and Parker [3, Corollary 5.3], it is related to the minimum volume of quaternionic hyperbolic orbifolds. More precisely, the minimum volume of a 2-dimensional quaternionic hyperbolic manifold with k cusps is $\dfrac{\sqrt{2}k}{15I_2} = \dfrac{\sqrt{2}k}{720}$.

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2. Criteria for existence of lifting

Let $\Pi \subset \mathbf{H}_7(\mathbb{H}) \rtimes \mathrm{Aut}(\mathbf{H}_7(\mathbb{H}))$ be an AB-group. Then $\Gamma = \Pi \cap \mathbf{H}_7(\mathbb{H})$ is a lattice and $\Phi = \Pi/\Gamma$ is finite. Since $\mathbb{R}^3$ is the center of the nilpotent group $\mathbf{H}_7(\mathbb{H})$, we have $\Gamma \cap \mathbb{R}^3 \cong \mathbb{Z}^3$ and $\Gamma/\mathbb{Z}^3 \cong \mathbb{Z}^4$. Moreover

$$1 \to \mathbb{Z}^3 \to \Pi \to Q \to 1$$

and

$$1 \to \mathbb{Z}^4 \to Q \to \Phi \to 1$$

are exact, and Q is naturally a 4-dimensional crystallographic group.

Construction with Q. For each Q (4-dimensional crystallographic group), we shall check if there exists a construction with Q; that is, a torsion free $\Pi \subset \mathbf{H}_7(\mathbb{H}) \rtimes \mathrm{Aut}(\mathbf{H}_7(\mathbb{H}))$ fitting the short exact sequence

$$1 \longrightarrow \mathbb{Z}^3 \longrightarrow \Pi \longrightarrow Q \longrightarrow 1.$$

This is the key notion for our arguments and construction. We have a complete classification of 4-dimensional crystallographic groups (Q's in the above statement). We shall use the presentations of the 4-dimensional crystallographic groups given in the book [1]. The crystallographic groups will be called Q, and every Q has an explicit representation $Q \longrightarrow \mathbb{R}^4 \rtimes \mathrm{GL}(4,\mathbb{Z})$ in this book.

Proposition 2.1. (Structure of $\mathrm{Aut}(\mathbf{H}_7(\mathbb{H}))$)

$$\mathrm{Aut}(\mathbf{H}_7(\mathbb{H})) \cong \mathrm{Hom}(\mathbb{H},\mathbb{R}^3) \rtimes O(\mathbf{J},4)$$

where

(1) $O(\mathbf{J},4) = SO(4) \times \mathbb{R}^+ \subset \mathrm{GL}(4,\mathbb{R})$
(2) $\mathrm{Hom}(\mathbb{H},\mathbb{R}^3) \rtimes O(\mathbf{J},4)$ *acts on* $\mathbf{H}_7(\mathbb{H})$ *by*

$$(\eta, A)\cdot(s,\mathbf{x}) = (\hat{A}s + \eta(\mathbf{x}), A\mathbf{x}).$$

Corollary 2.2. *For a 4-dimensional crystallographic group Q to have a construction, its holonomy group Ψ must be in $SO(4)$. Therefore, if Ψ contains a matrix of determinant -1, there is no construction from Q.*

By this Corollary, the following groups are eliminated:

- <u>1152</u>: 33/16: $\det(G) = -1$.
- <u>576</u>: 33/14: $\det(F) = -1$.
- <u>384</u>: 32/21: $\det(F) = -1$.
- <u>288</u>: 30/13: $\det(D) = -1$.
- <u>240</u>: 31/07: $\det(C) = -1$.
- <u>192</u>: 32/18: $\det(E) = -1$, 32/19: $\det(F) = -1$.
- <u>144</u>: 23/11, 29/09, 30/11, 30/12: $\det(C) = -1$.
- <u>128</u>: 32/17: $\det(D) = -1$.
- <u>120</u>: 31/04, 31/05: $\det(A) = -1$.
- <u>96</u>: 20/22: $\det(D) = -1$, 25/11: $\det(E) = -1$.
- <u>72</u>: 22/11, 23/09, 23/10: $\det(B) = -1$,
  23/07, 29/06, 29/08: $\det(C) = -1$, 29/07: $\det(D) = -1$.
- <u>64</u>: 19/06 (C), 32/13 (D), 32/14 (D), 32/15 (C): $\det = -1$.

Lemma 2.3. *Any finite subgroup of Q is an extension of a cyclic group by an orientable 3-dimensional holonomy group $1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, where the* *cyclic group can be conjugated into the kernel S^3 of the canonical map $SO(4)(\subset O(\mathbf{J},4)) \to SO(3)$.*

By Lemma 2.3, we eliminate the following groups:

• <u>144</u>:

33/11: The non-abelian group $Q_0 = \langle AC, B\rangle$ of order 24
induces a holonomy Q_2 of order 12 or a non-cyclic $Q_0 = Q_1$.

• <u>72</u>:

30/07: The non-abelian group $\langle B, C^3\rangle$ of order 24 induces
a holonomy Q_2 of order 12 or a non-cyclic Q_1 of order 12,
30/08: $\langle AB^4, BC\rangle$ induces a non-commutative holonomy of order 6,
30/09: $\langle AB^4, C\rangle$ induces a non-commutative holonomy of order 6,
33/07: The non-abelian group $Q_0 = \langle AC, B\rangle$ of order 24
induces a holonomy Q_2 of order 12 or a non-cyclic $Q_0 = Q_1$.

• <u>64</u>:

32/12/01/004: $\langle t_1(a, A)^6(d, D), t_3^{-1}(a, A)^2(d, D), (b, B)\rangle \cong (\mathbb{Z}_2)^3$.

• <u>60</u>:

31/03/01: $\langle (c, C)(d, D), (c, C)^2(d, D)^2\rangle$
induces a non-commutative holonomy of order 10,
31/03/02: $\langle (b, B), (d, D)\rangle$
induces a non-commutative holonomy of order 10.

Lemma 2.4. *Q cannot contain a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

By Lemma 2.4, we eliminate the following groups:

• <u>576</u>:

33/15: $\langle A, B^2, F\rangle \cong (\mathbb{Z}_2)^3$.

• <u>288</u>:

33/13: $\langle BD^3, C, D^6\rangle \cong (\mathbb{Z}_2)^3$.

• <u>192</u>:

32/20/01, 03: $\langle (a, A)^2, (b, B), (c, C)\rangle \cong (\mathbb{Z}_2)^3$,
32/20/02: $\langle (a, A)^2(b, B), (a, A)(c, C), t_2(b, B)\rangle \cong (\mathbb{Z}_2)^3$.

• <u>144</u>:

30/10: $\langle A, C^3, D\rangle \cong (\mathbb{Z}_2)^3$.

• <u>120</u>:

31/06: $\langle A, B, D^5\rangle \cong (\mathbb{Z}_2)^3$.

• <u>96</u>:

32/16/01, 32/16/03: $\langle A^2, C, D\rangle \cong (\mathbb{Z}_2)^3$,
32/16/02 : $\langle t_2^{-1}t_4^{-1}(d,D), t_4^{-1}(c,D), t_1t_2^{-1}(a,A)^2\rangle \cong (\mathbb{Z}_2)^3$.

• <u>72</u>:
23/08, 29/05: $\langle A^3, B^3, C\rangle \cong (\mathbb{Z}_2)^3$.

• <u>64</u>:
32/12/01/002: $\langle t_3(a,A)^4(c,C), (b,B), (c,C)\rangle \cong (\mathbb{Z}_2)^3$,
32/12/01/003: $\langle t_2^{-1}(a,A)^4(b,B), (b,B), (d,D)\rangle \cong (\mathbb{Z}_2)^3$,
32/12/01/004: $\langle t_1(a,A)^6(d,D), t_3^{-1}(a,A)^2(d,D), (b,B)\rangle \cong (\mathbb{Z}_2)^3$,
32/12/02/002: $\langle (a,A)^4, (b,B)(c,C), t_2^{-1}(c,C)\rangle \cong (\mathbb{Z}_2)^3$,
32/12/02/003: $\langle t_3^{-1}(a,A)^2(c,C), t_2^{-1}(a,A)^6(c,C),$
$t_2t_3^{-2}(a,A)^6(b,B)(c,C)\rangle \cong (\mathbb{Z}_2)^3$,
32/12/02/005: $\langle t_2(a,A)^6(d,D), t_4^{-1}(a,A)^2(d,D), t_2(b,B)\rangle \cong (\mathbb{Z}_2)^3$,
32/12/02/006: $\langle t_1^{-1}t_2^2(a,A)^6(d,D), t_2t_3(a,A)^4(b,B), t_2(b,B)\rangle \cong (\mathbb{Z}_2)^3$.

Lemma 2.5. *Suppose Q contains a finite subgroup Q_0 so that Q_0 fits an exact sequence $1 \to Q_1 \to Q_0 \to Q_2 \to 1$, where Q_1 induces a trivial action on $\mathbb{Z}^3$ and Q_2 induces an effective action on $\mathbb{Z}^3$.*

(1) *If $Q_1 \cong \mathbb{Z}_4$ and $Q_2 \cong \mathbb{Z}_6$, then $Q_0 \cong \mathbb{Z}_{24}$.*
(2) *If $Q_1 \cong \mathbb{Z}_6$ and $Q_2 \cong \mathbb{Z}_4$, then $Q_0 \cong \mathbb{Z}_{24}$ or $\mathbb{Z}_2 \times \mathbb{Z}_{12}$.*

Corollary 2.6. *There is no AB-group $\Pi \subset \mathbf{H}_7(\mathbb{H}) \rtimes \mathrm{Aut}(\mathbf{H}_7(\mathbb{H}))$ from Q given in* **33/10** (*holonomy group of order 96*).

Recall that any almost Bieberbach group Π for $\mathbf{H}_7(\mathbb{H})$ is a torsion free extension of $\mathbb{Z}_3$ by a 4-dimensional crystallographic group Q. There are 4783 4-dimensional crystallographic groups up to isomorphism. However, all 4-dimensional crystallographic groups are not qualified here. By Corollary 2.2, we eliminate about half of those 4-dimensional crystallographic groups. By Lemmas 2.3, 2.4, 2.5 and Corollary 2.6, we can eliminate most of the remaining 4-dimensional crystallographic groups, except the cases **33/08**, **33/09**, **33/12**, and the 4-dimensional crystallographic groups with holonomy group of order ≤ 48.

3. Elimination of special cases and the Existence with holinomy order 48

Given a 4-dimensional crystallographic group Q, it may happen that there is no construction from Q itself, but there may one from a conjugate of Q.

Remark 3.1. Two affinely conjugate representations of the same group into E(4) as crystallographic groups can result two non-isomorphic constructions.

Proposition 3.2. *Let $Q \subset \mathbb{R}^4 \rtimes SO(4)$ is a crystallographic group with holonomy group Φ. Suppose:*

If a symmetric matrix commutes with every element of Φ, then it is λI with $\lambda > 0$.

Then there is a construction $\Pi \subset \mathbf{H}_7(\mathbb{H}) \rtimes \mathrm{Aut}(\mathbf{H}_7(\mathbb{H}))$ from some embedding Q into $\mathbb{R}^4 \rtimes SO(4)$ if and only if there is a construction from Q or $(\mathbf{0},\sigma)^{-1}Q(\mathbf{0},\sigma)$.

Proposition 3.3. *Let $Q \subset \mathbb{R}^4 \rtimes SO(4)$ be a crystallographic group generated by*

$$(\mathbf{v}_1, I), (\mathbf{v}_2, I), (\mathbf{v}_3, I), (\mathbf{v}_4, I), (\mathbf{a}_1, A_1), (\mathbf{a}_2, A_2), \cdots, (\mathbf{a}_p, A_p),$$

where the subgroup $\langle (\mathbf{v}_1, I), (\mathbf{v}_2, I), (\mathbf{v}_3, I), (\mathbf{v}_4, I) \rangle$ is the maximal normal free abelian subgroup of rank 4*. Then there exists an* AB*-group obtained from Q if and only if the group $\Pi_{(s_1, s_2, \cdots, s_p)}$ generated by*

$$(0, \mathbf{v}_1, I), \cdots, (0, \mathbf{v}_4, I), (s_1, \mathbf{a}_1, A_1), \cdots, (s_p, \mathbf{a}_p, A_p),$$

(*for some $s_1, s_2, \cdots, s_p \in \mathbb{R}^3$*) *is an* AB*-group.*

Propositions 3.2 and 3.3 make it possible to eliminate the cases **33/08** and **33/09**. It turns out that the group given in **33/12** contains the group given in **33/08** as a normal subgroup of index 2. Thus there is no construction from the group given in **33/12**.

Proposition 3.4. *There is no construction $\Pi \subset \mathbf{H}_7(\mathbb{H}) \rtimes \mathrm{Aut}(\mathbf{H}_7(\mathbb{H}))$ from Q given in* **33/08** *or* **33/09** (*holonomy group of order 96*)*, or* **33/12** (*holonomy group of order* 192).

Now we construct an almost Bieberbach group whose holonomy group has order 48. This proves that the maximal order of the holonomy groups of all infra-nilmanifolds with $\mathbf{H}_7(\mathbb{H})$-geometry is 48.

Theorem 3.5 (Existence)**.** *There is an AB-group $\Pi \subset \mathbf{H}_7(\mathbb{H}) \rtimes \mathrm{Aut}(\mathbf{H}_7(\mathbb{H}))$ from Q given in* **33/05/01/003** (*holonomy group of order 48*).

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