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CAYLEY HYPERSURFACE AND FILIFORM LEFT SYMMETRIC ALGEBRA

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Abstract. We will find the relationship between nondegenerate homogeneous affine hypersurfaces which are graphs of functions and the complete abelian left symmetric algebra structures with the Hessian type metric. With this relationship, we will prove the Eastwood and Ezhov's conjecture on Cayley hypersurface which is the counter part of the abelian filiform left symmetric algebra.

1. Introduction

The Cayley surface in 3-dimensional affine space is given by

$$z = xy - \frac{1}{3}x^3.$$

It is equiaffinely homogeneous and moreover it is the orbit of the 2-dimensional abelian Lie group[NS94]. The Cayley surface has a remarkable property that its cubic form C is nonzero and parallel(*see* [NS94] for more), that is, $C \neq 0$ and $\nabla C = 0$, where ∇ is the induced affine connection. In fact, it is the only surface satisfying this condition up to affine congurant in $\mathbb{R}^3$ [NS94]. The cubic form property of the Cayley surface was studied for higher dimension in [DV91, DVY94].

In [EE00], M. Eastwood and V. Ezhov construct the *Cayley hypersurface* as a generalization of the Cayley surface which is unique up to affine congurent in each dimension : the 0-level surface of the following function in $(n+1)$-dimensional affine space

$$\Phi(x_1, \cdots, x_{n+1}) = \sum_{d=1}^{n+1} (-1)^d \frac{1}{d} \sum_{i+j+\cdots+m=n+1} \overbrace{x_i x_j \cdots x_m}^{d}. \tag{1.1}$$

The following Proposition is the results of Eastwood and Ezhov :

Proposition 1.1. [EE00] *Let Σ be an n-dimensional Cayley hypersurface. Then we have following* :

(a) *The affine automorphism group contains a transitive abelian subgroup.*
(b) *The isotropy subgroup of affine automorphism group is of 1-dimensional.*
(c) *The affine normals of Σ are everywhere parallel to the last coordinate.*

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They also conjectured the converse of this Proposition :

Conjecture 1. [EE00] *Let Σ be a non-degenerate hypersurface in $(n+1)$-dimensional affine space such that*

E1 Σ *admits a transitive abelian group $\mathbb{A}$ of affine motions.*
E2 $\mathrm{Aut}(\Sigma)$ *has an 1-dimensional isotropy group.*
E3 *Affine normals to Σ are everywhere parallel.*

Then Σ is a Cayley hypersurface.

For the nondegenerate hypersurface, if the affine normal ξ is constant and hence the shape operator $S = 0$, then with the Gauss equation

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY,$$

the induced connection ∇ is flat. In this case, after changing coordinates so that $\xi = (0, \cdots, 0, 1)$, Σ is given as a graph of a function, $x_{n+1} = F(x_1, \cdots, x_n)$. The author does not know whether the function F must be defined for all $\mathbb{R}^n$. Instead we will investigate the automorphism group of Σ and give a proof of Eastwood and Ezhov's conjecture with one more assumption that the domain of the function F is $\mathbb{R}^n$, that is, Σ is a graph over $\mathbb{R}^n$.

Note that the Cayley surface is a graph of a polynomial, and these polynomials will be called *Cayley polynomial.* Here is some example of Cayley polynomials :

$$\begin{aligned} x_3 &= x_1x_2 - \tfrac{1}{3}x_1^3. \\ x_4 &= x_1x_3 + \tfrac{1}{2}x_2^2 - x_1^2x_2 + \tfrac{1}{4}x_1^4 \\ x_5 &= x_1x_4 + x_2x_3 - x_1^2x_3 - x_1x_2^2 + x_1^3x_2 - \tfrac{1}{5}x_1^5. \end{aligned} \tag{1.2}$$

A vector space $\mathfrak{L}$ with a bilinear product which satisfy the left symmetric condition, for all $x, y, z \in \mathfrak{L}$

$$x(yz) - (xy)z = y(xz) - (yx)z,$$

is called *left symmetric algebra*(abbreviated as LSA). The associated Lie algebra $\mathfrak{L}^-$ of an LSA $\mathfrak{L}$ is obtained by defining the Lie bracket as following : for all $x, y \in \mathfrak{L}$

$$[x, y] = xy - yx.$$

From the left symmetric condition, this bracket satisfies the Jacobi identity, that is, it yields a Lie algebra structure. We will denote this Lie algebra as $\mathfrak{L}^-$. For a given LSA $\mathfrak{L}$, any $x, y \in \mathfrak{L}$, we denote λ_x and ρ_y the left and right multiplication in $\mathfrak{L}$, that is,

$$\lambda_x(y) = xy = \rho_y(x).$$

Proposition 1.2. [Kim96] *Let G be an n-dimensional simply connected Lie group and let $\mathfrak{g}$ be the Lie algebra of G. Then TFAE :*

(a) *There is a compatible LSA structure on $\mathfrak{g}$.*
(b) *G admits a left-invariant affine structure(torsion free and flat).*
(c) *There is a Lie group homomorphism* $\phi : G \to \mathrm{Aff}(n)$ *such that $d(ev_x)|_e$ is an isomorphism for some $x \in \mathbb{R}^n$.*
(d) *There is a Lie algebra homomorphism*

$$\varphi = (f, q) : \mathfrak{g} \to \mathrm{aff}(\mathbb{R}^n) = \mathrm{gl}(\mathbb{R}^n) + \mathbb{R}^n$$

such that q is a linear isomorphism.

(e) *There is a Lie algebra homomorphism from $\mathfrak{g}$ into $\mathrm{aff}(\mathfrak{g})$ of the form $x \mapsto (\lambda_x, x)$ with $\lambda_{[x,y]} = [\lambda_x, \lambda_y]$ and $\lambda_x y - \lambda_y x = [x, y]$.*

An LSA $\mathfrak{L}$ is called *complete* if the developing image of $\exp(\mathfrak{L}^-) = G$ is the whole $\mathbb{R}^n$.

Proposition 1.3. [Kim96] *For an LSA $\mathfrak{L}$, TFAE :*
(a) $\mathfrak{L}$ *is complete.*
(b) $\det(I + \rho_x) = 1$ *for all* $x \in \mathfrak{L}$.
(c) $\mathrm{tr}\rho$ *is a trivial function of* $\mathfrak{L}$.

For any LSA $\mathfrak{L}$, we have two(left, right) descending sequences of subalgebras of $\mathfrak{L}$ defined by

$$\mathfrak{L}^{i+1} = \mathfrak{L}\mathfrak{L}^i, \qquad \mathfrak{L}_{i+1} = \mathfrak{L}_i\mathfrak{L}.$$

With these two sequences, K. Dekimpe and V. Ongenae(1999) define a *filiform LSA* as an LSA $\mathfrak{L}$ satisfying :

$$\dim \mathfrak{L}^i = \dim \mathfrak{L}_i = n + 1 - i \quad \text{ for all } i \in \{1, \cdots, n\}.$$

Proposition 1.4. [DO99] *Let $\mathfrak{L}$ be an n-dimensional filiform LSA. Then,*
(a) *The associated Lie algebra $\mathfrak{L}^-$ is nilpotent.*
(b) $\mathfrak{L}$ *is a complete LSA.*
(c) $\mathfrak{L}^i$ *is equal to* $\mathfrak{L}_i$ *for all* $i \in \{1, \cdots, n\}$.
(d) $\mathfrak{L}$ *has a one-dimensional center* $c(\mathfrak{L}) = \mathfrak{L}^n = \mathfrak{L}_n$
(e) *For all* $i \in \{1, \cdots, n\}$, $\mathfrak{L}/\mathfrak{L}^i$ *is a filiform LSA.*
(f) *There is a adequate basis* $\{e_1, \cdots, e_n\}$ *satisfying*

$$e_1 e_j = e_{j+1} \quad \textit{for all } j \in \{1, 2, \cdots, n-1\}.$$

The adequate basis in Proposition 1.4 (f) is called *strongly adequate.*

2. Homogeneous graph

Let Σ be a nondegenerate affine hypersurface in $\mathbb{R}^{n+1}$, defined as a graph of a function $F : \mathbb{R}^n \to \mathbb{R}$ such that $F(0) = 0$, $dF_0 = 0$, $|DdF| = 1$ and the affine normal is given by $\xi = (0, \cdots, 0, 1)$.

2.1. **Automorphism group.** Let $\mathrm{Aut}(\Sigma)$ be the group of affine automorphism of Σ.

Lemma 2.1. *Any element g of* $\mathrm{Aut}(\Sigma)$ *leave invariant the induced connection and the direction of the affine normal ξ.*

Corollary 2.2. *The affine automorphism $g \in \mathrm{Aut}(\Sigma)$ is represented as following* :

$$g = \left(\begin{pmatrix} A & 0 \\ c^t & t \end{pmatrix}, \begin{pmatrix} a \\ F(a) \end{pmatrix} \right) \in \mathrm{Aut}(\Sigma), \tag{2.1}$$

where $A \in \mathrm{GL}(n)$ *and* $c, a \in \mathbb{R}^n$.

From the action of $g \in \mathrm{Aut}(\Sigma)$ on the point $(b, F(b))^t \in \Sigma$, by using the same notation as Corollary 2.2, $g \cdot (b, F(b))^t = (Ab + a,\ c^t b + tF(b) + F(a))^t \in \Sigma$. So we have the following equation

$$F(Ab + a) = c^t b + tF(b) + F(a). \tag{2.2}$$

Differenciating the above equation (2.2) gives us the following equations

$$dF_{Ab+a}A = c^t + tdF_b \tag{2.3}$$

$$DdF_{Ab+a}(A\cdot, A\cdot) = tDdF_b(\cdot, \cdot). \tag{2.4}$$

Since $|DdF| \equiv 1$, (2.4) says that $\det(A)^2 = t^n$. Therefore we have :

$$t = (\det(A))^{\frac{2}{n}}.$$

Define a map from the automorphism group

$$q : \mathrm{Aut}(\Sigma) \to \mathrm{Aff}(n), \quad g \mapsto (A, a)$$

then q is a group homomorphism, moreover it is a monomorphism. Let us assume that $\mathrm{Aut}(\Sigma)$ contains an abelian subgroup $\mathbb{A}$ which acts transitively on Σ.

Proposition 2.3. *The action of the abelian subgroup $\mathbb{A}$ on Σ is simply transitive.*

Proposition 2.3 says that the abelian subgroup $\mathbb{A}$ is connected, moreover we notice that $\mathbb{A}$ is the maximal abelian subgroup from the above Proof of the Proposition. The image of the abelian subgroup $\mathbb{A}$ by q, $\bar{\mathbb{A}} = q(\mathbb{A})$ which is an n dimensional subgroup of $\mathrm{Aff}(n)$, acts simply transitively on $\mathbb{R}^n$. So we obtain the following :

(1) the standard flat affine connection on $\mathbb{R}^n$ induces the flat left invariant affine connection on $\bar{\mathbb{A}}$,

(2) the connection on $\bar{\mathbb{A}}$ defines a complete abelian LSA structure on the Lie algebra $\bar{\mathcal{A}} = \mathrm{Lie}\ \bar{\mathbb{A}}$,

(3) the left multiplication λ_a is nilpotent for all $a \in \bar{\mathcal{A}}$, therefore $\exp \lambda_a$ is unipotent.

Furthermore we have following:

Proposition 2.4. *For any element $g \in \mathbb{A}$, there exists a unique $a \in \mathbb{R}^n$ such that $g = g_a$ where g_a is represented by*

$$g_a = (\begin{pmatrix} M_a & 0 \\ c_a^t & 1 \end{pmatrix}, \begin{pmatrix} a \\ F(a) \end{pmatrix})$$

where $c_a \in \mathbb{R}^n$ and $M_a \in \mathrm{GL}(n)$ is the unipotent matrix given by the equations : for $\alpha \in \bar{\mathcal{A}}$

$$M_a = \exp \lambda_\alpha, \quad \textit{and} \quad a = \text{“}e^\alpha - 1\text{”} \in \mathbb{R}^n,$$

with $\text{“}e^\alpha - 1\text{”} = \alpha + \dfrac{1}{2!}\alpha^2 + \cdots$. *Moreover M_a acts on $\mathbb{R}^n$ as an isometry with respect to the Hessian metric DdF.*

Proposition 2.5. *Let $\bar{\mathbb{A}} = \{(M_a, a) | a \in \mathbb{R}^n\} \subset \mathrm{Aff}(n, \mathbb{R})$ be an abelian affine subgroup which acts simply transitively on $\mathbb{R}^n$. Then the subspace $\mathcal{N} = \{(N_a, a) | (M_a, a) \in \bar{\mathbb{A}}, N_a = M_a - I\}$ defines a complete abelian LSA structure on $\mathbb{R}^n$ defined by $a * b := N_a b$.*

Now, let us consider the Lie algebra of the abelian subgroup $\mathbb{A}$. Put $\mathcal{A}^- = \mathrm{Lie}\ \mathbb{A}$.

Lemma 2.6. *Any element of $\mathcal{A}^- = Lie\ \mathbb{A}$ is represented as the following*

$$l_a := \Big(\begin{pmatrix} N_a & 0 \\ a^t H & 0 \end{pmatrix}, \begin{pmatrix} a \\ 0 \end{pmatrix} \Big),$$

for some $a \in \mathbb{R}^n$.

Theorem 2.7. *If the* $\mathrm{Aut}(\Sigma)$ *contains a transitive abelian subgroup, then the function F is the polynomial of the form :*

$$F(\text{“}e^a - 1\text{”}) = a^t H (\frac{a}{2!} + \frac{a^2}{3!} + \cdots).$$

Theorem 2.8. *If an element g of $\mathrm{GL}(n+1)$ normalizes the abelian subgroup $\mathbb{A}$, then g is contained in the isotropy subgroup J. Moreover the following are equivalent* :

(a) $g = \begin{pmatrix} A & 0 \\ 0 & t \end{pmatrix} \in \mathrm{GL}(n+1)$ *normalizes* $\mathbb{A}$.

(b) $A \in \mathrm{Aut}(\mathbb{R}^n, *)$ *and* $F(Aa) = tF(a)$ *where* $t = (\det A)^{\frac{2}{n}}$.

(c) $A \in \mathrm{Aut}(\mathbb{R}^n, *)$ *and* $A^t H A = tH$ *where* $t = (\det A)^{\frac{2}{n}}$.

2.2. **Abelian filiform LSA.**

Lemma 2.9. *Let $\mathcal{A}$ be an abelian filiform LSA with the strongly adequate basis $\{e_1, \cdots, e_n\}$, then we have following* :

(a) *The following metric $<,>$ is nondegenerate Hessian type* :

$$< e_i, e_j >= \begin{cases} 1 & i+j = n+1 \\ 0 & i+j \neq n+1 \end{cases}$$

(b) *The matrix B,which is given by $Be_i = \dfrac{i}{n+1} e_i$, is a derivation of $\mathcal{A}$ and it also an infinitesimal similarity with respect to the metric $<,>$.*

Theorem 2.10. *The LSA $\mathcal{A}$ is abelian filiform if and only if the characteristic polynomial of $\mathcal{A}$ is the Cayley polynomial.*

3. Eastwood and Ezhov conjecture

3.1. **LSA version of the conjecture.** Now we restate the Conjecture of Eastwood and Ezhov in terms of LSA with the conditions that Σ is a graph of a function from $\mathbb{R}^n$ and $\mathrm{Aut}(\Sigma)$ acts transitively on the domain Ω lying over Σ.

Conjecture 2 (LSA version)**.** *Let $\mathcal{A}$ be a complete abelian LSA with nondegenerate Hessian type metric. Suppose that the group of the similarity LSA automorphism of $\mathcal{A}$ with respect to the Hessian type metric, $\mathrm{Aut}(\mathcal{A}) \cap \mathrm{Sim}(\mathcal{A})$, is of dimension 1, then $\mathcal{A}$ is the filiform LSA.*

Proposition 3.1. *Under the assumption that Σ is a graph of a function from $\mathbb{R}^n$ into $\mathbb{R}$ and the automorphism of Σ is not all of determinant 1, the Eastwood and Ezhov conjecture and its LSA version are equivalent.*

3.2. **Proof of the conjecture.**

Proposition 3.2. *Let $\mathcal{A}$ be a complete abelian LSA with a nondegenerate Hessian type metric $<,>$, then $\mathcal{A}$ is a filiform if and only if* $\dim c(\mathcal{A}) = 1$.

Proposition 3.3. *If the dimension of $c(\mathcal{A})$ is lager than 1, there exists a nontrivial automorphism which is an isometry of the Hessian type metric.*

Proof of the Conjecture. Let $\mathcal{A}$ be a complete abelian LSA with nondegenerate Hessian type metric. If $\mathcal{A}$ is not the filiform LSA, then from Proposition 3.2, $\dim c(\mathcal{A}) \geq 2$. Therefore, from Proposition 3.3, $\mathcal{A}$ has a nontrivial automorphism which is an isometry of the Hessian type metric. Moreover, from the assumption that the automorphism of Σ is not all of determinant 1, we have, at least, 1-dimension similarity which is an automorphism of $\mathcal{A}$ and not isometry. This says that $\dim \mathrm{Aut}(\mathcal{A}) \cap \mathrm{Sim}(\mathcal{A}) \geq 2$. □

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