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SMOOTH CHARACTERIZATION OF SYMPLECTIC $\mathbb{C}P^n$

DOSANG JOE

Abstract. Let X be a compact $2n$ (≥ 6) dimensional symplectic manifolds. Suppose X contains a *simple* rational curve $C \in X$ with $-K_X \cdot C = n+1$ and there is at least one curve $C_{p,q}$ passing through the given points $p, q \in X$. Then X can be shown to be diffeomorphic to $\mathbb{C}P^n$. The condition above can be rephrased in terms of Gromov-Witten invariants which also characterizes the smooth structure of $\mathbb{C}P^n$. This result leads to the problem of uniqueness of symplectic structure of $\mathbb{C}P^n$ and the generalization of Gromov-McDuff theorem which is a characterization of $\mathbb{C}P^2$.

1. Introduction

In early 90's, Y. Miyaoka announced a result about the characterization of the projective space in terms of rational curves. However the proof of the theorem was incomplete in defining the mini rational tangent variety of X, called MRTV. With a long collaboration with K. Cho, N.I. Shephard-Barron, a proof of the theorem was given [1]. Six month later, S. Kebekus [2, 3] published a couple of papers which analyzed the singular rational curves, in which he completed the following theorem with much shorter proof and substantial technical simplification.

Theorem 1.1 (Cho, Miyaoka, Shephard-Barron and Kebekus)**.** *Let X be a projective manifold of dimension $n \geq 3$, defined over $\mathbb{C}$. Assume that for every curve $C \subset X$, we have $-K_X \cdot C \geq n+1$. Then X is isomorphic to the projective space.*

The purpose of this article is to prove the symplectic version of theorem 1.1. The history in the perspective of development of symplectic world should begin with the early work of M. Gromov [6]. In this paper, he introduced J-holomorphic curves in the symplectic manifolds which generalizes the curves in algebraic varieties. After his work, there have been many mathematicians including D.Mcduff, D.Salamon, G.Tian and K.Fukaya (1986-1996) who put great efforts on the foundation of Quantum cohomology and Gromov-Witten theory. This leads to a proof of the Arnold's Conjecture as a result.

2. Algebraic vs. Symplectic

In the proof of Theorem 1.1, they make use of the Mori's bend and break lemma [4] which is a fundamental technique developed for the birational classification of higher dimensional algebraic varieties (so called Mori's program). For instance, with the assumption of Theorem 1.1, it can be proved that the "minimal" rational

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curve $C \cong \mathbf{P}^1 \subset X$ has the normal bundle $N_{\mathbf{P}^1/X} \cong \oplus_1^{n-1} O_{\mathbf{P}^1}(1)$ which allows one to show that the mini rational tangent map (the generalized Gauss map) is surjective. This technical lemma draws a line between algebraic and symplectic category. There has been no known result so far in the literature about the "Mori's bend & break technique" for general symplectic manifolds. We don't even know that there is a canonical symplectic structure in the moduli space of *simple* rational curves. Nevertheless, there is freedom to vary the almost complex structure tamed to the given symplectic form ω so that we can achieve the transversality theorem for the moduli space of stable curves of genus 0. Via this genericity theorem for the general symplectic manifolds, we can show that there is no cuspidal rational curve for the general J for example. This leads to the following main theorem we are going to prove.

3. Main theorem

Theorem 3.1 (Main)**.** *Let X be a simply connected compact symplectic manifold of dimension $2n \geq 6$. Assume that there exit* simple *symplectically immersed rational curves $C_{p,q}$ passing through any two given point $p, q \in X$, $-K_X(H) = n+1$, $H = [C_{p,q}] \in H_2(X, \mathbb{Z})$. Then X is diffeomorphic to $\mathbb{C}P^n$.*

The notion of *simple* class of rational curves corresponds to the minimality of the rational curve class in algebraic category which indicates that there is no reducible component of the stable curves among the moduli space of rational curves. This leads to the compactness of the given moduli space of stable curves corresponding to the class $H \in H_2(X, \mathbb{Z})$. The condition $-K_X(H) = n+1$ is equal to the one in Theorem 1.1. The inequality in Theorem 1.1 turns into the equality with minimal condition in algebraic case. The assumption on the "...passing through any two given point $p, q \in X$..." due to the lack of techniques for the Mori's bend & break lemma for the symplectic manifolds. We put the "...immersed rational curve..." term for the sake of simplicity. Applying the genericity lemma, we can reduce the condition to the one involving the Gromov-Witten invariants.

Let us introduce some notations from Gromov-Witten theory. $X_{0,k,H}$ denotes the moduli space of pseudo-holomorphic rational curves of k-marked point which represent the homology class $H \in H_2(X, \mathbb{Z})$. The complex dimension of $X_{0,k,H}$ is $-K_X(H) + n - 3 + k$. Let H be the simple homology class which represent the pseudo-holomorphic curves $C_{p,q}$. Then the moduli space $X_{0,k,H}$ is compact of complex dimension $2n + k - 2$ due to a result of Gromov [6]. Let $ev_i : X_{0,k,H} \to X$ be the i-th evaluation map and $\pi : X_{0,k,H} \to X_{0,k-1,H}$ be the forgetful map of k-th marked point.

Corollary 3.2. *Let X be a simply connected compact symplectic manifold of dimension $2n \geq 6$. Assume that there exists a simple pseudo-holomorphic rational curve C such that $[C] = H \in H_2(X, \mathbb{Z})$ and $\int_{X_{0,2,H}} ev_1^*\mu \wedge ev_2^*\mu \neq 0$ where $<\mu>= H^{2n}(X, \mathbb{Z})$. Then X is diffeomorphic to $\mathbb{C}P^n$.*

In the proof of the corollary, we do need to have the following proposition which restores the assumption of the main theorem.

Proposition 3.3. *For generic J, the moduli space $X^J_{0,k,H}$ is a smooth compact manifold of dimension $4n + 2k - 4$. Moreover for each $[\mathbf{P}^1, f, x_1, \cdots x_k] \in X_{J,k,H}$, the induced J-holomorphic map $f : \mathbf{P}^1 \to (X, \omega, J)$ defines an immersion.*

Here we use the notation for the moduli of stable J-holomorphic curve with k marked points of genus 0 to $X^J_{0,k,H}$. Let us briefly sketch the proof of the corollary. It just follows from the enumerative interpretation of primary Gromov-Witten invariant, i.e. non-vanishing two point correlation function $\int_{X_{0,2,H}} ev_1^*\mu \wedge ev_2^*\mu \neq 0$ indicates that there is an immersed rational curve passing through any given two point $p, q \in X$. This argument restores the assumption of main theorem.

4. Conjectures and Discussions

Upon the consequence of main theorem, it gives rise to some problems concerning the smooth and symplectic structure of $\mathbb{C}P^n$.

Conjecture 4.1. *Let X be a simply connected compact symplectic manifold. Assume that there exits a* simple *pseudo-holomorphic rational curve C such that $-K_X \cdot C = n+1$. Then X is diffeomorphic to $\mathbb{C}P^n$.*

The conjecture 4.1 can be fixed if one can establish the symplectic structure on $X_{0,k,H}$ and symplectic version of Mori's bend & break technique.

Conjecture 4.2. *The symplecture structure of $\mathbb{C}P^n$ is unique.*

The second conjecture is the generalization of "Gromov-Mcduff-Taubes" theorem [6, 7, 5] which proves uniqueness of symplectic structure of symplectic $\mathbb{C}P^2$. It might be interesting to construct the exotic symplectic structure on $\mathbb{C}P^n$ by calculating the two pointed Gromov-Witten invariants such as $\int_{X_{0,2,H}} ev_1^*\mu \wedge ev_2^*\mu \neq 1$, which disprove the conjecture.

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Department of Mathematics, Pohang University of Science and Technology, Pohang, Kyungbuk, Korea 790-784

E-mail address: joe@euclid.postech.ac.kr