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WEAKLY LAGRANGIAN AND NORMAL EMBEDDING OF SPHERES INTO $\mathbb{C}^n$

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ABSTRACT. We provide complete answers to the questions: (a) Which S^n admits a weakly Lagrangian or normal embedding into $\mathbb{C}^n$? (b) When the product of spheres admits a weakly Lagrangian or normal embedding into the complex Euclidean plane?

0. INTRODUCTION

The notions of totally real embeding, weakly Lagrangian embedding, and normal embedding are weaker versions of Lagrangian embedding.

The notion of normal submanifold was introduced by J. C. Sikorav([S]) as a weaker version of Lagrangian submanifold.

And Polterovich([P]) showed that if L is a closed normal non-Lagrangian submanifold of a symplectic manifold M and the Euler characteristic of L vanishes then its displacement energy $e(L)$ vanishes.

And the notion of weakly Lagrangian embedding was introduced by T. Kawashima ([K]). He showed that S^n admits a weakly Lagrangian embedding into $\mathbb{C}^n$ if and only if $n = 1, 3$, from which it follows that S^n does not admit any Lagrangian embedding into $\mathbb{C}^n$ if $n \neq 1, 3$. In fact, later it has been shown that, for any manifold M^n which admits a Lagrangian embedding into $\mathbb{C}^n$, we have $\pi_1(M) \neq 1$([G]). Therefore it follows that S^n admits a Lagrangian embedding into $\mathbb{C}^n$ only when $n = 1$.

Our basic questions are: (a) Which S^n admits a weakly Lagrangian (normal) embedding into $\mathbb{C}^n$? (b) When the product of spheres admits a weakly Lagrangian (normal) embedding into the complex Euclidean plane?

In the first and second sections, we answer the questions as for the weakly Lagrangian embedding and normal embedding, respectively.

The basic notions such as 'normal', 'symplectic', 'weakly Lagrangian' etc. are explained in Sections 1 and 2 below.

1. WEAKLY LAGRANGIAN EMBEDDING

Two subbundles η_0 and η_1 of a vector bundle ξ over a topological space M is said to be *homotopic* if there exists a subbundle $\tilde{\eta}$ of $\xi \times I$ such that $\tilde{\eta}|_{M\times\{0\}} = \eta_0$ and $\tilde{\eta}|_{M\times\{1\}} = \eta_1$.

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A *symplectic form* on a vector bundle is a nondegenerate two form on it. A vector bundle of finite rank is referred to as a *symplectic vector bundle* if it is considered with a fixed symplectic two form. Note that a symplectic vector bundle should be of even rank. A subbundle η of a symplectic vector bundle ξ is a *Lagrangian subbundle* if 2(rank η) = rank ξ and the restriction of the symplectic form to η is the zero form. A subbundle η of a symplectic vector bundle ξ is called a *weakly Lagrangian subbundle* if η is homotopic to a Lagrangian subbundle of ξ.

Now let $f : L \to M$ be an embedding (resp. immersion) of a smooth manifold L into a symplectic manifold M with a symplectic structure ω. We call f a *Lagrangian embedding* (resp. *immersion*) if the tangent bundle TL of L is a Lagrangian subbundle of the symplectic vector bundle f^*TM (with the symplectic form $f^*\omega$). Similarly, f is a *weakly Lagrangian embedding* (resp. *immersion*) if TL is a weakly Lagrangian subbundle of f^*TM.

We will consider $\mathbb{C}^n$ with the usual symplectic structure. A Lagrangian embedding or a weakly Lagrangian embedding will be understood as 'into $\mathbb{C}^n$' unless otherwise specified.

Note that the notion of weakly Lagrangian embedding (resp. immersion) is invariant under regular homotopy. That is, if f_0 and f_1 are embeddings (resp. immersions) homotopic through immersions and f_0 is a weakly Lagrangian embedding (resp. immersion), then f_1 is also such.

Furthermore we have $TL = (-1)^{\frac{n(n-1)}{2}}\nu(f)$ for a weakly Lagrangian immersion f of L into a symplectic manifold M. On the other hand, we recall that every embedding of S^n into $\mathbb{C}^n$ is regularly homotopic to the standard one(cf. [Smale]). Thus we conclude that in the case there exists a weakly Lagrangian embedding of S^n into $\mathbb{C}^n$, the standard embedding ι is weakly Lagrangian, and hence satisfies the condition $TS^n = (-1)^{\frac{n(n-1)}{2}}\nu(\iota)$. Clearly the normal bundle of the standard embedding of S^n into $\mathbb{C}^n$ is trivial. Thus in order that there exists a weakly Lagrangian embedding of S^n into $\mathbb{C}^n$, TS^n should be trivial and eventually $n \neq 1, 3, 7$ is not in the case. Here making use of the homotopy theory, we reach the following conclusion.

Theorem 1.1 [K]. *S^n admits al weakly lagrangian embedding into $\mathbb{C}^n$ if and only if $n = 1, 3$.*

An immersion $f : M^m \to P^{2m}$ is referred to as *completely regular* if it has no triple points, that is, $|f^{-1}y| \leq 2$ for any $y \in P$, and is self-transverse. If $f : M \to P$ is a completely regular immersion, one may define the intersection number $I(f)$ of f as follows: (i) For the mod 2 intersection number, one defines $I(f) \in \mathbb{Z}_2$ as the number of the double points mod 2. (ii) Assume that M, P are oriented and m is even. Then one may define the integral intersection number as follows: Let $r = f(p) = f(p')$, $p \neq p'$, be a double point of f. Let $v = (v_1, v_2, \cdots, v_m)$, $v' = (v'_1, v'_2, \cdots, v'_m)$ be sequences of tangent vectors which represent the orientation of M at p and p', respectively. If the sequence of tangent vectors $(dfv, dfv') = (dfv_1, dfv_2, \cdots, dfv_m, dfv'_1, dfv'_2, \cdots, dfv'_m)$ represents the orientation of P at r, write $\varepsilon_r = +1$ and, otherwise, write $\varepsilon_r = -1$. Note that ε_r remains unchanged even if we interchange p, p'. Define $I(f) = \sum_r \varepsilon_r \in \mathbb{Z}$, where r runs through all the double points of f.

If f, g are completely regular immersions which are regularly homotopic to each other, then we have $I(f) = I(g)$: According to J. Cerf([C]), for generic regular homotopy, the double points vary continuously except at a finite set of points at each of which a pair of double points appear or disappear. If m is even, the two have opposite values for ε_r. Furthermore, since every immersion is regularly homotopic to a completely regular immersion, it follows that $I(f)$ is well-defined for any immersion f.

Now assume $m \geq 3$ and P is simply connected. Let $I(f)$ denote the mod 2 intersection number if the dimension of M is odd or M is unorientable and, in the remaining case, the integral intersection number. Then $I(f)$ vanishes if and only if the regular homotopy class of f can be represented by an embedding, which is a consequence of the Whitney trick(cf. [BY3]).

Lemma 1.2 [BY3]. *Let $f : M^m \to P^{2m}$, $g : N^n \to Q^{2n}$ be immersions where M, N are closed smooth manifolds and P, Q, smooth manifolds. Then, for the mod 2 intersection numbers, we have*

$$I(f \times g) = \chi(\nu_f)I(g) + I(f)\chi(\nu_g) \in \mathbb{Z}_2 \ ,$$

where $\chi(\cdot)$ is the Euler characteristic in $\mathbb{Z}_2$-coefficients. Furthermore, assume M, N, P, Q are oriented, and both m, n are odd. Then, for the integral intersection numbers, we have

$$I(f \times g) = 0 \in \mathbb{Z} \ .$$

And let M, N be smooth closed manifolds of dimension m and n, $m, n \geq 1$, respectively. And let $f : M^m \to \mathbb{C}^m$, $g : N^n \to \mathbb{C}^n$ be weakly Lagrangian immersions. If m and n are odd, then, by Lemma 1.2 above, we have $I(f \times g) = 0$. This implies that $f \times g$ is regularly homotopic to an embedding. Now since being weakly Lagrangian is invariant under regular homotopy, $M \times N$ admits a weakly Lagrangian embedding into $\mathbb{C}^{m+n}$.

If m is odd, n is even, then $\chi(M) = 0$. Furthermore, by assumption $\chi(N)$ is an even integer. Now note that $\nu_f \cong (-1)^{m(m-1)/2}TM$, $\nu_g \cong (-1)^{n(n-1)/2}TN$ (see [BY1]). Therefore, $I(f \times g) = \chi(\nu_f)I(g) + I(f)\chi(\nu_g) = 0 \in \mathbb{Z}_2$. We conclude that $f \times g$ is regularly homotopic to an embedding. Thus $M \times N$ admits a weakly Lagrangian embedding into $\mathbb{C}^{m+n}$. Thus we have

Theorem 1.3 [BY2]. *Let M, N be smooth closed manifolds of dimension m and n, $m, n \geq 1$, respectively. Assume M, N admit weakly Lagrangian immersions into $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. If both m, n are odd, then $M \times N$ admits a weakly Lagrangian embedding into $\mathbb{C}^{m+n}$. In the case that m is odd and n is even, we assume further that $\chi(N)$ is an even integer. Then $M \times N$ admits a weakly Lagrangian embedding into $\mathbb{C}^{m+n}$.*

And also we have:

Corollary 1.4. *$S^{n_1} \times S^{n_2}$, $n_i \geq 1, i = 1, 2$, admits a weakly Lagrangian embedding into $\mathbb{C}^{n_1+n_2}$ if and only if some n_i is odd.*

Proof. Assume n_1 is odd without loss of generality. Since each S^{n_i}, $i = 1, 2$, admits a Lagrangian immersion into $\mathbb{C}^{n_i}$, we may apply Theorem 1.3 to conclude that $S^{n_1} \times S^{n_2}$ admits a weakly Lagrangian embedding into $\mathbb{C}^{n_1+n_2}$.

On the other hand, if all n_i are even, then $\chi(S^{n_1} \times S^{n_2}) \neq 0$. Thus $S^{n_1} \times S^{n_2}$ does not admit any weakly Lagrangian embedding into $\mathbb{C}^{n_1+n_2}$(see [BY1]). □

2. Normal embedding

Let M be a symplectic manifold. Note that dim M is even. Let L be a smooth manifold of dimension $\frac{1}{2}dim\, M$ and let $f : L \to M$ be an embedding (resp. immersion) of L into a symplectic manifold M with a symplectic structure ω.

We call f a *normal embedding* (resp. *immersion*) if there is a Lagrangian subbundle $\mathbb{L}$ of f^*TM which is transverse to TL. Note that every Lagrangian submanifold L of M is normal.

We will consider $\mathbb{C}^n$ with the usual symplectic structure. A Lagrangian embedding or normal embedding must be understood as 'into $\mathbb{C}^n$' unless otherwise specified.

First of all we need the following.

Lemma 2.1 [BY4]. *Let f be a normal embedding of a smooth oriented n-dimensional manifold L into a symplectic $2n$-dimensional manifold M. Then*

$$TL \cong \nu_f$$

where TL is the tangent bundle of L and ν_f, the normal bundle of f.

And we have the following.

Corollary 2.2. *If a smooth oriented closed n-manifold L admits a normal embedding into $\mathbb{C}^n$, then we have*

$$\chi(L) = 0$$

where $\chi(L)$ is the Euler number of L.

Proof. Regard L as a normal submanifold of $\mathbb{C}^n$ and let ν denote the normal bundle. Consider the normal neighborhood N of L. Then one of the generator U of

$$H^n(D\nu, S\nu; \mathbb{Z}) \cong H^n(N, \partial N : \mathbb{Z}) \cong Z$$

pulled back to $H^n(N;\mathbb{Z}) \cong H^n(L;\mathbb{Z})$ is the Euler class of TL, presuming a suitable orientation of L, since $\nu \cong TL$ by Lemma 2.1 above. The Euler class evaluated at the fundamental class of L is the Euler number of L. However U when pulled back to $H^n(N)$ is the zero element for we have the following commutative diagram:

$$\begin{array}{ccc} H^n(\mathbb{C}^n, \mathbb{C}^n - \text{int}N; \mathbb{Z}) & \longrightarrow & H^n(N, \partial N; \mathbb{Z}) \\ \downarrow & & \downarrow \\ H^n(\mathbb{C}^n; \mathbb{Z}) & \longrightarrow & H^n(N; \mathbb{Z}) \end{array}$$

where all the arrows come from the inclusions. □

As for the relation between normal embedding and weakly Lagrangian embedding, we have the following.

Theorem 2.3 [BY4]. *Let M be a symplectic $2n$-manifold and L be a smooth n-manifold which admits a normal embedding into M. If L is parallelizable, then the embedding is weakly Lagrangian.*

And we have the following.

Corollary 2.4. *S^7 does not admit any normal embedding into $\mathbb{C}^7$.*

Proof. Assume that S^7 admits a normal embedding into $\mathbb{C}^7$. Then, since S^7 is parallelizable, the normal embedding is weakly Lagrangian by Theorem 2.3 above. But according to Kawashima([K]), S^n admits a weakly Lagrangian embedding if and only if $n = 1, 3$. This means that S^7 does not admit any normal embedding. □

If $f : S^n \to \mathbb{C}^n$ is an embedding, then the normal bundle ν_f of f must be trivial since it is stably trivial and its Euler characteristic vanishes. So by Lemma 2.1 the tangent bundle TS^n is trivial. Thus if $n \neq 1, 3, 7$, S^n does not admit any normal embedding into $\mathbb{C}^n$.

On the other hand, S^1 admits a Lagrangian embedding. Also by applying an observation of Polterovich([P]), S^3 has a normal embedding since S^3 admits totally real embedding and it is parallelizable.

Thus we have the following.

Theorem 2.5. *S^n admits a normal embedding into $\mathbb{C}^n$ if and only if n is 1 or 3.*

Remark. (i) A totally real submanifold L of a symplectic manifold which is parallelizable is normal according to Polterovich. Theorem 2.3 further means that L is weakly Lagrangian.

(ii) Note that Theorem 2.3 together with an easy explicit construction of the Lagrangian subbundle transverse to TS^3 for the standard embedding of S^3 into $\mathbb{C}^3$ proves that the standard embedding is weakly Lagrangian(cf. [K]).

Now let us consider the normal embedding of product of spheres.

If both m and n are even, then $\chi(S^m \times S^n) \neq 0$, by Corollary 2.2, $S^m \times S^n$ does not admit any normal embedding.

If m or n is odd, then $S^m \times S^n$ admits a totally real embedding into $\mathbb{C}^{m+n}$(cf. Theorem 1,[SZ]) and $S^m \times S^n$ is parallelizable. Therfore, according to Polterovich([P]), $S^m \times S^n$ admits a normal embedding into $\mathbb{C}^{m+n}$.

Thus we have the following.

Theorem 2.6. *$S^m \times S^n$, $m, n \geq 1$, admits a normal embedding into $\mathbb{C}^{m+n}$ if and only if m or n is odd.*

Note that H. Hofer([H]) defined the *displacement energy* of a subset A of a symplectic manifold M as

$$\inf\{\operatorname*{Max}_{M\times I} H - \operatorname*{Min}_{M\times I} H | H \in \mathcal{C} \text{ such that } g_H^1 A \cap A = \emptyset\}$$

where $\mathcal{C}$ is the set of all smooth real valued functions on $M \times I$ which attain both maximum and minimum and g_H^1 is the Hamiltonian flow at time 1 determined by H.

The normal embedding of Theorem 2.6 is not necessarily the product of the standard embeddings(see [SZ]) and therefore their images may not be contained in a codimension 1 plane. Also the embedding is not Lagrangian unless some n_i is 1. Even if some n_i is 1 and the embedding is Lagrangian, we recall the fact that any Lagrangian embedding of a manifold of dimension greater than 1 can be C^l-approximated for any $l \geq 1$, by non-Lagrangian normal embeddings([P]). Therefore, Theorem 1.2 in [P] by L. Polterovich implies:

Corollary 2.7. *Assume $k > 1$. If some n_i, $i = 1, 2, \cdots, k$, is odd, the product of spheres, $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$, $n_i \geq 1$, $i = 1, 2, \cdots, k$, $k \geq 2$, admits a normal embedding into $\mathbb{C}^{n_1+n_2+\cdots+n_k}$, for which the displacement energy vanishes.*

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