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DEHN FILLINGS ON 3-MANIFOLDS

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Abstract. In this article we give a survey of the results that are known on Dehn surgeries on knots in S^3 and the distances between Dehn fillings on hyperbolic 3-manifolds creating essential small surfaces.

1. Introduction

The Dehn surgery construction is a way of obtaining a closed 3-manifold as the following: remove a solid torus neighborhood $N(K)$ of some knot K in the 3-sphere S^3 and sew it back differently. In particular, Dehn showed that, taking K to be the trefoil, one could obtain infinitely many non-simply-connected homology spheres in this way. The *exterior* of K, denoted by $E(K) = S^3 - \mathrm{int}N(K)$, has the torus boundary $\partial E(K)$. If γ is an isotopy class of a simple closed curve on $\partial E(K)$ that bounds a meridional disk in the reattached solid torus, we denote the resulting closed orientable 3-manifold by $K(\gamma)$, and say that it is obtained by γ*-Dehn surgery on* K.

More generally, one can consider the manifolds $L(\vec{\gamma})$ obtained by $\vec{\gamma}$-Dehn surgery on an l-component link $L = K_1 \cup \cdots \cup K_l$ in S^3, where $\vec{\gamma} = (\gamma_1, \ldots, \gamma_l)$ is an l-tuple of surgery coefficients. Lickorish [21] and independently Wallace [27] proved that every closed orientable 3-manifold can be constructed in this way. Thus a good understanding of Dehn surgery might lead to progress on general questions about the structures of 3-manifolds.

Starting with the case of knots, it is natural to extend the context a little in the following way. Let M be a compact, connected and orientable 3-manifold with a torus boundary component $\partial_0 M$. Let γ be a *slope* on $\partial_0 M$, that is, the isotopy class of an essential simple closed curve on $\partial_0 M$. The 3-manifold obtained from M by γ*-Dehn filling* is defined to be $M(\gamma) = M \cup V_\gamma$, where V_γ is a solid torus glued to M along $\partial_0 M$ in such a way that γ bounds a meridian disk in V_γ. If γ_1 and γ_2 are two slopes on $\partial_0 M$, the *distance* $\Delta(\gamma_1, \gamma_2)$ denotes their minimal geometric intersection number.

In this article we give a survey of the results that are known on Dehn surgeries on knots in S^3 and the distances between Dehn fillings on hyperbolic 3-manifolds creating essential small surfaces.

2. Dehn surgeries on knots in the 3-sphere

Let K be a knot in the 3-sphere S^3. Recall that a slope is an isotopy class of an essential simple closed curve on the torus $\partial E(K)$. Slopes on $\partial E(K)$ can be

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parametrized by $\mathbb{Q}\cup\{\infty\}$, using meridian-longitude basis $\{\mu,\lambda\}$ for $H_1(\partial E(K))$ so that $\gamma \leftrightarrow p/q$ if and only if $[\gamma] = p\mu + q\lambda$ in $H_1(\partial E(K))$. Here, ∞ represents the meridional slope, and the distance between two slopes p/q and r/s is given by the formula $\Delta(p/q, r/s) = |ps - qr|$.

The focus in Dehn surgery theory may be indeed the following question. Which 3-manifolds can be obtained from Dehn surgeries on knots? For example, Dehn surgeries on the trivial knot yield S^3, $S^1 \times S^2$ and Lens spaces. What can we say about nontrivial Dehn surgeries on nontrivial knots? From now on we assume that knots are nontrivial.

For any knot K, $K(\infty)$ is the 3-sphere. It is then natural to consider whether or not a nontrivial Dehn surgery can produce S^3. Gordon and Luecke gave the following negative answer.

Theorem 2.1. [9] *Every nontrivial Dehn surgery on a knot never yields the* 3-*sphere.*

The above theorem implies that knots are determined by their complements, that is, given two knots K and K' having homeomorphic complements, there exists homeomorphism $h : S^3 \to S^3$ such that $h(K) = K'$. This Gordon and Luecke's theorem was a solution of the long-standing knot complement problem. It is worthy to compare this theorem with the following so-called Property P Conjecture, which is still open, although it is known to be true for many classes of knots.

Conjecture 2.2. *Every nontrivial Dehn surgery on a knot yields a non-simply-connected* 3-*manifold.*

Now we turn our attention to Dehn surgeries yielding a manifold with Heegaard genus one, i.e., $S^1 \times S^2$ or lens spaces.

For homological reasons, only 0-surgery can yield $S^1 \times S^2$. However, using taut foliation Gabai showed that this cannot happen.

Theorem 2.3. [6] $K(0) \neq S^1 \times S^2$.

Regarding lens spaces, we have the following conjecture. Compare with Property P Conjecture.

Conjecture 2.4. $K(\gamma)$ *is neither* $L(2,1), L(3,1)$ *nor* $L(4,1)$.

A knot K in S^3 is said to be *strongly invertible* if there is an orientation preserving involution of S^3 which leaves K invariant and reverses the orientation of K. The involution has two fixed points on K. Then Hirasawa and Shimokawa gave a partial solution of Conjecture 2.4.

Theorem 2.5. [18] *No Dehn surgery on a strongly invertible knots can yield the lens space* $L(p,1)$ *for any even integer* p.

Which knots admit lens space surgery? And do we have restrictions on surgery coefficients for lens space surgery? For example, if K is the $(-2,3,7)$ pretzel knot, which is strongly invertible, $K(18) = L(18,5)$ and $K(19) = L(19,7)$. First, we investigate the restriction on coefficients.

Theorem 2.6. [4, Cyclic Surgery Theorem] *When* K *is not a torus knot, if* $K(\gamma)$ *is a lens space, then* γ *must be an integer.*

A knot is a *torus knot* if it lies on a standardly embedded torus in S^3. Remark that Dehn surgeries on tours knots are classified by Moser [23].

Now consider which knots admit lens space surgery. This is known to be very difficult problem. However, Berge [1] gives an explicit construction for knots admitting lens space surgeries as the following.

Let K be a knot lying on a genus two Heegaard surface F for $S^3 = H_1 \cup_F H_2$. Then K is said to be *doubly primitive* if $[K]$ is part of a basis for two generator free group $\pi_1(H_i)$ for each $i = 1, 2$. $\partial N(K) \cap F$ consists of two essential loops on $\partial N(K)$, which determine a slope γ. Then one can find that $K(\gamma)$ is a lens space. Berge listed doubly primitive knots.

Conjecture 2.7. *If $K(\gamma)$ is a lens space for some γ, then K appears in Berge's list.*

Remark that an affirmative answer to this conjecture would imply that Conjecture 2.4 is also true.

Now we turn to the creation of essential spheres and tori via Dehn surgery on a knot. Recall that a sphere in a 3-manifold M is *essential* if it does not bound a 3-ball in M, and that a properly embedded surface, not a sphere, is *essential* if it is incompressible, ∂-incompressible and non-∂-parallel in M.

K is called a *cable knot* if it is an essential loop on a torus $\partial N(K_0)$ for some knot K_0, running at least twice longitudinally. Two curves $\partial N(K) \cap \partial N(K_0)$ determine a slope γ as above. Then $K(\gamma)$ contains an essential sphere. The following Cabling Conjecture asserts that this is only way of creating essential spheres under Dehn surgeries on knots.

Conjecture 2.8. *If $K(\gamma)$ contains an essential sphere, then K is a cable knot.*

This conjecture is known for satellite knots [25], alternating knots [22], knots with bridge number no greater than five [19] and symmetric knots [16].

Regarding essential tori, Gordon and Luecke proved the following.

Theorem 2.9. [10] *If $K(\gamma)$ contains an essential torus, then $\Delta(\gamma, \infty) \leq 2$. Moreover, if $\Delta = 2$, K is a strongly invertible knot.*

Finally, we consider a very nice method attacking Conjectures 2.2, 2.7 and 2.8 simultaneously. This is to use essential lamination theory. See [7] for the definition of essential lamination. A knot K is *persistently laminar* if $E(K)$ contains an essential lamination λ and λ remains essential in $K(\gamma)$ for any nontrivial slope γ. It is a result in [7] that if a 3-manifold contains an essential lamination, then it has a universal cover homeomorphic to $\mathbb{R}^3$. Thus $K(\gamma)$ contains no essential sphere and has an infinite fundamental group.

Theorem 2.10. [2, 3, 17, 20] *Many knots are persistently laminar.*

3. Dehn fillings creating essential small surfaces

By a *small surface* we mean one with non-negative Euler characteristic, i.e., a sphere, disk , annulus or torus. Such surfaces play a special role in the theory of 3-manifolds. Thurston's geometrization theorem for Haken manifolds [26] asserts that a hyperbolic 3-manifold M with non-empty boundary contains no essential small surfaces. Furthermore, if M is hyperbolic, then the Dehn filling $M(\gamma)$ is also hyperbolic for all but finitely many slopes [26], and a good deal of attention has been directed towards obtaining a more precise quantification of this statement.

Let us say that a 3-manifold is of *type* $\mathcal{S}, \mathcal{D}, \mathcal{A}$ or $\mathcal{T}$, if it contains an essential sphere, disk, annulus or torus. The bound $\Delta(X_1, X_2)$ is the least number n such that if M is a hyperbolic manifold which admits two Dehn fillings $M(\gamma_1), M(\gamma_2)$ of type X_1, X_2, respectively, then $\Delta(\gamma_1, \gamma_2) \leq n$. The numbers are known in all cases, and are summarized in the following table.

	$\mathcal{S}$	$\mathcal{D}$	$\mathcal{A}$	$\mathcal{T}$
$\mathcal{S}$	1	0	2	3
$\mathcal{D}$		1	2	2
$\mathcal{A}$			5	5
$\mathcal{T}$				8

The upper bounds in various cases are due to the following. $(\mathcal{S}, \mathcal{S})$: Gordon and Luecke [11]; $(\mathcal{S}, \mathcal{D})$: Scharlemman [25]; $(\mathcal{S}, \mathcal{A})$: Wu [30]; $(\mathcal{S}, \mathcal{T})$: Oh [24] and Wu [29]; $(\mathcal{D}, \mathcal{D})$: Wu [28]; $(\mathcal{D}, \mathcal{A})$: Gordon and Wu [14]; $(\mathcal{D}, \mathcal{T})$: Gordon and Luecke [12]; $(\mathcal{A}, \mathcal{A})$: Gordon and Wu [13]; $(\mathcal{A}, \mathcal{T})$: Gordon and Wu [15]; $(\mathcal{T}, \mathcal{T})$: Gordon [8].

See also [5] for examples realizing the maximal distances.

References

[1] J. Berge, *Some knots with surgeries yielding lens spaces*, preprint.

[2] M. Brittenaham, *Persistently laminar tangles*, J. Knot Theory Ramifications **8** (1999), 415–428.

[3] M. Brittenaham, *Persistent laminations from Seifert surfaces*, J. Knot Theory Ramifications **10** (2001), 1155–1168.

[4] M. Culler, C. McA. Gordon, J. Luecke, and P.B. Shalen, *Dehn surgery on knots*, Ann. of Math. **125** (1987), 237–300.

[5] M. Eudave-Muñoz and Y.Q. Wu, *Nonhyperbolic Dehn fillings on hyperbolic 3-manifolds*, Pacific J.Math. **190** (1999), 261–275.

[6] D. Gabai, *Foliations and the topology of 3-manifolds. III*, J. Diff. Geom. **26** (1987) 479–536.

[7] D. Gabai and U. Oetel, *Essential laminations in 3-manifolds*, Ann. of Math. **130** (1989), 41–73.

[8] C. McA. Gordon, *Boundary slopes on punctured tori in 3-manifolds*, Trans. Amer. Math. Soc. **350** (1998), 1713–1790.

[9] C. McA. Gordon and J. Luecke, *Knots are determined by their complements*, J. Amer. Math. Soc. **2** (1989), 371–415.

[10] C. McA. Gordon and J. Luecke, *Dehn surgeries on knots creating essential tori, I*, Communications in Analysis and Geometry **3** (1995), 597–644.

[11] C. McA. Gordon and J. Luecke, *Reducible manifolds and Dehn surgery*, Topology **35** (1996), 385–409.

[12] C. McA. Gordon and J. Luecke, *Toroidal and boundary-reducing Dehn fillings*, Topology Appl. **93** (1999), 77–90.

[13] C. McA. Gordon and Y.Q. Wu, *Annular Dehn fillings*, Comment. Math. Helv. **75** (2000), 430–456.

[14] C. McA. Gordon and Y.Q. Wu, *Annular and boundary reducing Dehn fillings*, Topology **39** (2000), 531–548.

[15] C. McA. Gordon and Y.Q. Wu, *Toroidal and annular Dehn fillings*, Proc. London Math. Soc. **78** (1999), 662–700.

[16] C. Hayashi and K. Shimokawa, *Symmetric knots satisfy the cabling conjecture*, Math. Proc. Cambridge Philos. Soc. **123** (1998), 501–529.

[17] M. Hirasawa and T. Kobayashi, *Pre-taut sutured manifolds and essential laminations*, Osaka J. Math. **38** (2001), 905–922.

[18] M. Hirasawa and K. Shimokawa, *Dehn surgeries on strongly invertible knots which yield lens spaces*, Proc. A.M.S. **128** (2000) 3445–3451.

[19] J. Hoffman, *Reducing spheres and embedded projective planes after Dehn surgery on a knot*, Ph.D Thesis, The university of Texas at Austin, (1995).

[20] S. Lee and S. Oh, *Constructing persistently laminar knots*, Topology Appl. **124** (2002), 139–143.

[21] W.B.R. Lickorish, *A representation of orientable combinatorial* 3*-manifolds*, Ann. of Math. **76** (1962), 531–540.

[22] W. W. Menasco and M. B. Thistlethwaite, *surfaces with boundary in alternating knot exteriors*, J. Reine Angew. Math. **426** (1992) 47–65.

[23] L. Moser, *Elementary surgery along a torus knot*, Pacific J. Math. **38** (1971) 734–745.

[24] S. Oh, *Reducible and toroidal* 3*-manifolds obtained by Dehn fillings*, Topology Appl. **75** (1997), 93–104.

[25] M. Scharlemann, *Producing reducible* 3*-manifolds by surgery on a knot*, Topology **29** (1990), 481–500.

[26] W. Thurston, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. **6** (1982), 357–381.

[27] A. Wallace, *Modifications and cobounding manifolds*, Canad. J. Math. **12** (1960) 503–528.

[28] Y.Q. Wu, *Incompressibility of surfaces in surgered* 3*-manifolds*, Topology **31** (1992), 271–279.

[29] Y.Q. Wu, *Dehn fillings producing reducible manifolds and toroidal manifolds*, Topology **37** (1998), 95–108.

[30] Y.Q. Wu, *Sutured manifold hierarchies, essential laminations, and Dehn surgery*, J. Diff. Geom. **48** (1998), 407–437.

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