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TWO MULTIPLICATIONS IN INFINITE BRAID-PERMUTATION GROUP

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Abstract. The braid-permutation group BP_n of rank n, which is a special kind of Artin group, is a group of welded braids. It is also described as a subgroup of the automorphism group of a free group. The disjoint union of BP_n's forms a symmetric monoidal category $\mathcal{BP}$ so that the group completion of the classifying space of $\mathcal{BP}$ is an infinite loop space. Since the abelianization of BP_n equals $\mathbb{Z} \oplus \mathbb{Z}/2$ we can conjecture that

$$BBP^+ \simeq S^1 \times B\Sigma_\infty^+ \times Y,$$

for some infinite loop space Y. V. Vershinin claimed that he constructed of a splitting map, though a significant gap has been found. In order to fill the gap we should verify a certain compatibility of two distinct multiplications on BBP^+.

1. Introduction

Since the classical expression of braid groups was given by Artin ([1], [2]) many definitions and interpretations of braids have been found. In a geometric point of view, braids arise as isotopy classes of a collection of n connected strings in three-dimensional space. A braid diagram may be thought of as a composite of two types of *crossings* of strings (Figure 2.1). A welded braid diagram is obtained from the composite of these crossings and the *welded crossings* (Figure 2.2). The set of welded braids forms a group, called the braid-permutation group BP_n(cf.[5], [6]). It was shown by R. Fenn, Rimányi and Rourke([5], [6]) that BP_n is also given by the set of generators $\{\xi_i, \sigma_i \mid i = 1, 2, \ldots, n-1\}$ and three types of relations: braid group relations, symmetric group relations, mixed relations. This expression of BP_n is analogous to the classical group presentation of the braid group given by Artin. As a matter of fact, BP_n is a special kind of Artin group which recently attracts interest of many mathematicians (cf. [3], [4]).

The group BP_n may also be regarded as a subgroup of the automorphism group $AutF_n$ of a free group on $\{x_1, \ldots, x_n\}$. The generator σ_i of BP_n, called the braid group generator, is given by

$$\begin{cases} x_i & \mapsto x_{i+1} \\ x_{i+1} & \mapsto x_{i+1}^{-1} x_i x_{i+1} \\ x_j & \mapsto x_j, j \neq i, i+1 \end{cases} .$$

The generator ξ_i, called the symmetric group generator, is given by

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$$\begin{cases} x_i & \mapsto x_{i+1} \\ x_{i+1} & \mapsto x_i \\ x_j & \mapsto x_j, j \neq i, i+1 \end{cases} .$$

The braid group B_n and the symmetric group Σ_n are naturally embedded in BP_n.

One of the interesting properties of BP_n is that the subgroup PC_n of $AutF_n$ of the automorphisms of permutation-conjugacy type is isomorphic to BP_n. Moreover, BP_n is isomorphic to the automorphism group $AutFQ_n$ of the free quandle of rank n, and is closed related to the automorphism group $AutFR_n$ of the free rack of rank n(cf. [6]) and these groups have relations to the invariants of classical knots and links in the 3-sphere.

The disjoint union of braid-permutation groups $\coprod_{n\geq 0} BP_n$ forms a symmetric monoidal category, denoted by $\mathcal{BP}$. The monoidal structure of $\mathcal{BP}$ is induced by the juxtaposition of braids:

$$BP_m \times BP_n \longrightarrow BP_{m+n}.$$

By the classical result of May ([7]), the group completion of the classifying space of $\mathcal{BP}$, which is known to be equivalent to $\Omega B(\coprod_{n\geq 0} BBP_n)$, has the homotopy type of an infinite loop space. The Quillen's plus construction of BBP is homotopy equivalent to the base point connected component of $B\mathcal{BP}$, that is,

$$BBP^+ \simeq \Omega_0 B(\coprod_{n\geq 0} BBP_n).$$

We can show that for $0 \leq n \leq \infty$ the abelianization of BP_n is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2$ (Lemma 3.1). From this we may conjecture the following splitting:

$$BBP^+ \simeq \Omega^\infty S^\infty \times S^1 \times Y$$

for some infinite loop space Y. Note that $\Omega^\infty S^\infty \simeq B\Sigma^+$ and $S^1 \simeq B\mathbb{Z}$.

There is an obvious map $p : BBP^+ \longrightarrow B\Sigma^+ \times S^1$ induced by obvious epimorphisms on the group level. Since BBP^+ is an H-space, in order to prove the above splitting of BBP^+ it suffice to show that there exists a splitting map (or a section) which is a right inverse of the map p. V. Vershinin constructed s splitting map, but it contains a certain gap. There are two different multiplication structures on BBP^+. One is the multiplication on the infinite loop space which is, in fact, induced by juxtaposing two braids. Another one is induced by the binary operation of *group* which is induced by stacking two braids. In the Vershinin's construction of the splitting map, these two multiplication structures were used in a confusing manner. The compatibility of two multiplications should be understood. This is a rather a classical and standard problem in a coherent theory of loop spaces.

2. The braid-permutation group

Let F_n be the free group of rank n with the set of generators $\{x_1, \ldots, x_n\}$, and let $AutF_n$ be the group of automorphisms of F_n. There are the standard inclusions of the symmetric group Σ_n and the braid group B_n into $AutF_n$. They can be described as follows:

Let $\xi_i \in AutF_n,\ i = 1, 2, \ldots, n-1$, be given by the following formula

$$(2.1)\qquad \begin{cases} x_i & \mapsto x_{i+1} \\ x_{i+1} & \mapsto x_i \\ x_j & \mapsto x_j, j \neq i, i+1 \quad . \end{cases}$$

Let $\sigma_i \in AutF_n,\ i = 1, 2, \ldots, n-1$, be given by the following formula

$$(2.2)\qquad \begin{cases} x_i & \mapsto x_{i+1} \\ x_{i+1} & \mapsto x_{i+1}^{-1} x_i x_{i+1} \\ x_j & \mapsto x_j, j \neq i, i+1 \quad . \end{cases}$$

Let BP_n be the subgroup of $AutF_n$ generated by ξ_i's and σ_i's of (2.1) and (2.2). BP_n is called the *braid-permutation group*. It was proved by R. Fenn, R. Rimányi and C. Rourke in [5], [6] that this group is given by the set of generators $\{\xi_i, \sigma_i \mid i = 1, 2, \ldots, n-1\}$ and the following relations:

The symmetric group relations,

$$\begin{cases} \xi_i^2 & = 1, \\ \xi_i \xi_j & = \xi_j \xi_i, \quad \text{if} \quad |i-j| > 1, \\ \xi_i \xi_{i+1} \xi_i & = \xi_{i+1} \xi_i \xi_{i+1}. \end{cases}$$

The braid group relations,

$$\begin{cases} \sigma_i \sigma_j & = \sigma_j \sigma_i, \ \text{if} \ |i-j| > 1, \\ \sigma_i \sigma_{i+1} \sigma_i & = \sigma_{i+1} \sigma_i \sigma_{i+1}. \end{cases}$$

The mixed relations,

$$\begin{cases} \sigma_i \xi_j & = \xi_j \sigma_i, \ \text{if} \ |i-j| > 1, \\ \xi_i \xi_{i+1} \sigma_i & = \sigma_{i+1} \xi_i \xi_{i+1}, \\ \sigma_i \sigma_{i+1} \xi_i & = \xi_{i+1} \sigma_i \sigma_{i+1}. \end{cases}$$

Fenn, Rimányi and Rourke also gave the geometrical interpretation of BP_n as a group of *welded braids*. First they defined a *welded braid diagram* on n strings as a collection of n monotone arcs starting from n points on a horizontal line of a plane (the top of the diagram) and going down to n points on another horizontal line (the bottom of the diagram). The diagrams can have crossings of two types: (A) ordinary braids in Figure 2.1; (B) welds in Figure 2.2.

Figure 2.1.

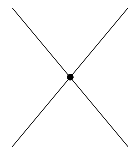

Figure 2.2.

An example of a welded braid diagram is shown in Figure 2.3.

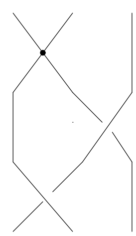

Figure 2.3.

Fenn, Rimányi and Rourke defined the following types of allowable transformations on welded braid diagrams. They are described in Figures 2.6, 2.7, 2.8. The transformations in Figure 2.7 are Reidemeister transformations of knot theory. The first transformation in Figure 2.8 corresponds to the relation

$$\xi_i^2 = 1 \quad .$$

The transformation in Figure 2.9 is the geometric form of the commutativity from the mixed relations. There are also analogous transformations corresponding to the commutativity from the symmetric group and the braid group relations.

A *welded braid* is defined as an equivalence class of welded braid diagrams under allowable transformations. It was proved by Fenn, Rimányi and Rourke that welded braids form a group, and this group is isomorphic to the braid-permutation group BP_n. The generator σ_i corresponds to the canonical generator of the braid group B_n and is shown in Figure 2.4.

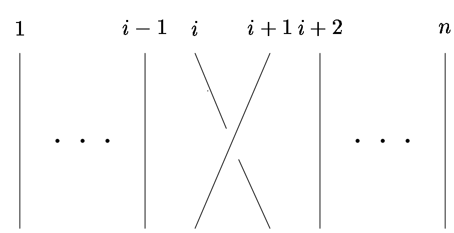

Figure 2.4.

The generators ξ_i correspond to the welded braids shown in the Figure 2.5.

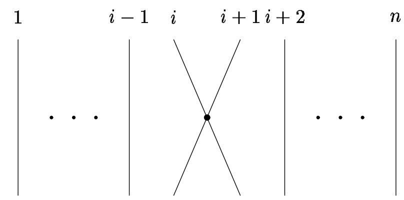

Figure 2.5.

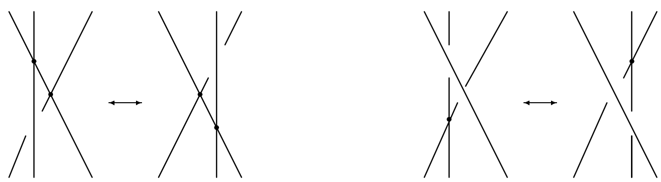

Figure 2.6.

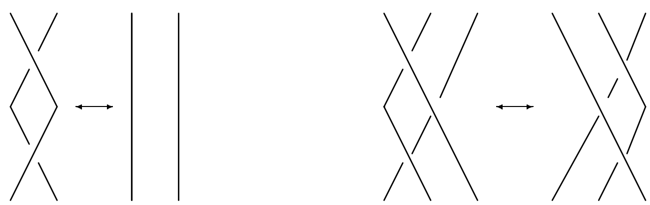

Figure 2.7.

Figure 2.8.

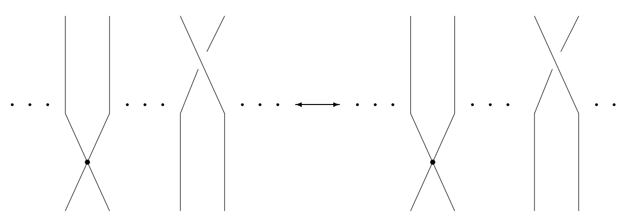

Figure 2.9.

3. Infinite loop space and a splitting conjecture

The collection of braid-permutation groups forms a symmetric monoidal category. We may regard the disjoint union of braid-permutation groups $\coprod_{n\geq 0} BP_n$ as a category whose objects are nonnegative integers $[0], [1], \cdots$ and morphism sets are as follows:

$$Hom([m],[n]) = \begin{cases} \emptyset \text{ if } m \neq n \\ BP_m \text{ if } m = n \end{cases}.$$

This category, denoted by $\mathcal{BP}$, is a symmetric monoidal category whose tensor product is induced by the juxtaposition of two welded braids:

$$BP_m \times BP_n \to BP_{m+n} \ .$$

Note that for each $n \in \{0, 1, \cdots\}$, BP_n contains the symmetric group Σ_n. The symmetric structure of $\mathcal{BP}$ is induced by the welded braid transposing the first m points and the last n points. By the classical result of Segal, the group completion of the classifying space of $\mathcal{BP}$ has the homotopy type of an infinite loop space.

By the theorem of May ([7]), the group completion of the classifying space of $\mathcal{BP}$ is homotopy equivalent to $\Omega B(\coprod_{n\geq 0} BBP_n)$. For the infinite braid-permutation group BP_∞, denoted simply by BP, we have

$$BBP^+ \simeq \Omega_0 B(\coprod_{n\geq 0} BBP_n) \ ,$$

where $+$ denotes the Quillen's plus construction and $\Omega_0 B(\coprod_{n\geq 0} BBP_n)$ means the connected component of $\Omega B(\coprod_{n\geq 0} BBP_n)$ containing the base point. BBP^+ is obtained by attaching 2 and 3-cells to BBP to kill the perfect commutator subgroup of BP(the fundamental group of BBP) without changing the homology.

Now we know that BBP^+ is an infinite loop space. We now investigate the splitting of this space into some well-known infinite loop spaces. Through the splitting we can get information on the homology of the stable braid-permutation group. The following lemma gives us a clue to the splitting of the space(cf. [9]).

Lemma 3.1. *For $2 \leq n \leq \infty$, the abelianization of BP_n is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2$, that is,*

$$BP_n/[BP_n, BP_n] \cong \mathbb{Z} \oplus \mathbb{Z}/2.$$

For a collection of groups G_n each of which contains the symmetric group Σ_n, $\coprod_{n\geq 0} G_n$ is a symmetric monoidal category so that the group completion of its classifying space is an infinite loop space and the following splitting lemma has been known([9]):

Lemma 3.2. *For an infinite loop space X, we have the following infinite loop space splitting:*

$$\Omega B(\coprod_{n\geq 0} BG_n) \simeq \Omega^\infty S^\infty \times X.$$

Note that $\Omega^\infty S^\infty \simeq B\Sigma_\infty^+$, $S^1 \simeq B\mathbb{Z}$. From Lemma 3.1 we can conjecture that $\Omega^\infty S^\infty$ and S^1 are the factors of the splitting.

Conjecture. *The group completion of the classifying space of $\mathcal{BP}$ is homotopy equivalent to*

$$\Omega^\infty S^\infty \times S^1 \times Y$$

for some infinite loop space Y.

Let j_n be the inclusion of the group $\mathbb{Z}$ into B_n:

$$j_n : \mathbb{Z} \to B_n,$$

where the generator of the cyclic group is mapped to one of the generators say, $j_n(1) = \sigma_1$.

There are epimorphisms

$$\alpha_n : BP_n \to \mathbb{Z},$$

$$\beta_n : BP_n \to \Sigma_n$$

which are given by the following formulas:

$$\alpha_n(\xi_i) = 0, \quad \alpha_n(\sigma_i) = 1 \quad \text{and} \quad \beta_n(\xi_i) = \xi_i, \quad \beta_n(\sigma_i) = \xi_i \quad \text{for all } i.$$

Let $\nu_n : \Sigma_n \to BP_n$ and $\kappa_n : B_n \to BP_n$ be inclusions.

The homomorphism α_n induces a functor of symmetric monoidal categories

$$\mathcal{BP} \to \mathcal{Z}$$

which induces the map of infinite loop spaces

$$\Omega B(\coprod_{n\geq 0} BBP_n) \to S^1.$$

Analogously, the homomorphism $\beta_n : BP_n \to \Sigma_n$ induces the map of infinite loop spaces

$$\Omega B(\coprod_{n\geq 0} BBP_n) \to \Omega^\infty S^\infty.$$

The maps α_n and β_n induce the following map ϕ:

$$\Omega B(\coprod_{n\geq 0} BBP_n) \to \Omega B(\coprod_{n\geq 0} BBP_n) \times \Omega B(\coprod_{n\geq 0} BBP_n) \to \Omega^\infty S^\infty \times S^1.$$

Since $\Omega B(\coprod_{n\geq 0} BBP_n)$ is an infinite loop space (so it is an H-space), by the splitting theorem of Wagoner ([10]) it suffices to construct a splitting map which means a homotopy right inverse of ϕ. The following is Vershinin's construction of the splitting map.

Let $q : X \to X^+$ be the plus construction map for a space X. Define $g : BB \to BBP^+$ by

$$g(b) = (qB\tau(b))^{-1} qB\kappa(b).$$

We also define

$$\psi : B\Sigma \times BB \overset{qB\nu\times g}{\longrightarrow} BBP^+ \times BBP^+ \overset{\mu}{\longrightarrow} BBP^+,$$

where μ is the multiplication map of a loop space. Then he claimed that the composite

$$B\Sigma \times S^1 \to B\Sigma \times BB \overset{\psi}{\longrightarrow} BBP^+$$

gives rise to the desired splitting map.

Vershinin's splitting map makes sense on the group level, which is expressed in a rough form as follows: Define

$$\Sigma \times \mathbb{Z} \overset{f}{\longrightarrow} BP$$

$$(x, n) \mapsto \kappa j(x) \xi_1^{-n} \sigma_1^n,$$

where κ and j are perviously defined inclusions. He probably claimed his proof because f is the right inverse of the group homomorphism

$$(\beta, \alpha) : BP \longrightarrow \Sigma \times \mathbb{Z}.$$

There are two different multiplication structures on BBP^+. One is the multiplication on the infinite loop space which is, in fact, induced by *juxtaposing* two braids. Another one is induced by the binary operation of *group* which is induced by *stacking* two braids. In the Vershinin's construction of the splitting map, these two multiplication structures were used in a confusing manner. The compatibility of two multiplications should be understood. More precisely speaking, we should verify that the following diagram commutes up to homotopy:

$$BB \longrightarrow BBP^+ \times BBP^+ \xrightarrow{a\times 1} BBP^+ \times BBP^+ \xrightarrow{\mu} BBP^+$$

$$\searrow_{(B(\nu\circ\kappa), B\kappa)} \qquad\qquad \nearrow_q$$

$$BBP \times BBP \xrightarrow{Bb\times 1} BBP \times BBP \xrightarrow{Bm} BBP$$

Here the maps a and b are induced by the inverse map $x \mapsto x^{-1}$ on the group level. The map $Bm : BBP \times BBP \to BBP$ is induced by the group multiplication map $m : BP \times BP \to BP$. The commutativity of the above diagram implies a certain compatibility of two multiplication structures on BBP^+.

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