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ENERGY DECAY FOR THE WAVE EQUATIONS IN EXTERIOR DOMAINS

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Abstract. We study the decay estimates of the energy for the wave equation in an exterior domain with a localized dissipation. The dissipative term consists of the following two parts: The first part may be nonlinear and localized in suitable bounded area, while the second part is linear in the outside of a big ball. So we may call such a dissipation as 'half-linear' dissipation.

1. Introduction

In this paper we consider the initial-boundary value problem for the wave equations with a dissipation ;

$$u_{tt} - \triangle u + \rho(x, u_t) = 0 \quad \text{in } \Omega \times (0, \infty) \tag{1.1}$$

$$u(x, 0) = u_0, \quad u_t(x, 0) = u_1 \qquad \text{in } \Omega \tag{1.2}$$

$$u(x, t) = 0 \qquad \text{on } \partial\Omega \times (0, \infty), \tag{1.3}$$

where Ω is an exterior domain in $\mathbb{R}^N$ such that $V \equiv \mathbb{R}^N \backslash \Omega$ is compact, the boundary $\partial\Omega$ is smooth, say C^3 class, and $\rho(x, v)$ is a function like $\rho(x, v) = a(x)g(v)$ with $a(x) \geq 0$ and $g'(v) \geq 0$. When $\Omega = \mathbb{R}^N$, Cauchy problem, the boundary condition (1.3) should be dropped.

To state our precise assumptions on $\rho(x, v)$, we introduce a part of the boundary $\partial\Omega$ as follows (see Lions [2]):

$$\Gamma(x_0) = \{x \in \partial\Omega | (x - x_0) \cdot \nu(x) > 0\},$$

where $x_0 \in \mathbb{R}^N$ is arbitrarily fixed and $\nu(x)$ denotes the outward unit normal vector at $x \in \partial\Omega$.

We set $B_r := \{x \in \mathbb{R}^N | |x| < r\}$ and $\Omega_r := \Omega \cap B_r$.

Let $a(x)$ be a nonnegative bounded function on Ω , satisfying:

(a) There exist $x_0 \in \mathbb{R}^N$ and a relatively open set ω in $\overline{\Omega}$ such that

$$\overline{\Gamma(x_0)} \subset \omega \text{ and } a(x) \geq \varepsilon_0 > 0 \ a.e. \text{ in } \omega; \tag{1.4}$$

(b) There exists $R > 0$ such that

$$a(x) \geq \varepsilon_0 > 0 a.e. \text{ for } |x| \geq L. \tag{1.5}$$

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With $a(x)$ above we now make the hypotheses on $\rho(x, v)$.

Hyp.A $\rho(x, v)$ is a monotone increasing and differentiable function in $v(\neq 0)$ and satisfies the following conditions:

$$\rho(x, v) = a(x)v \text{ if } (x, v) \in B_R^C \times \mathbb{R}, \tag{1.6}$$

for some $R > 0$. If $(x, v) \in \Omega_R \times \mathbb{R}$ and $|v| \leq 1$, then

$$k_0 a(x)|v|^{p+2} \leq \rho(x, v)v \leq k_1 a(x)\{|v|^{p+2} + |v|^2\}, \tag{1.7}$$

and if $(x, v) \in \Omega_R \times \mathbb{R}$ and $|v| \geq 1$, then

$$k_0 a(x)|v|^{q+2} \leq \rho(x, v)v \leq k_1 a(x)\{|v|^{q+2} + |v|^2\}, \tag{1.8}$$

where $k_0, k_1 > 0$.

We note that our dissipation may be localized near a part of the boundary $\Gamma(x_0)$ and near infinity and may vanish in a large area. The main purpose of this paper is to derive a decay estimate of the energy for the problem (1.1)-(1.3).

When $\rho(x, u_t) = a(x)u_t$,linear in the whole domain, decay properties of the solutions of (1.1)-(1.3) have been investigated by one of the present authors in [7] and it is an interesting problem to generalize them to the case of nonlinear dissipation like $\rho(x, u_t) = a(x)|u_t|^r u_t$. The problem with nonlinear localized dissipations in bounded domains has been investigated in Tcheugoué Tébou [9], Martinez [4] etc. and we can expect some fruitful results for unbounded domains. However, concerning the decay property of the wave equation with nonlinear dissipations in the whole space $\mathbb{R}^N$ very few results seem to be known. K. Mochizuki and T.Motai [5] treated the case $\rho(x, u_t) = |u_t|^r u_t, 0 < r \leq 2/(N-2)^+$, and proved a logarithmic decay estimate

$$E(t) \equiv \frac{1}{2}\int_{\mathbb{R}^N} (|u_t(t)|^2 + |\nabla u(t)|^2)dx \leq C\{log(1+t)\}^{-\nu}, \nu > 0.$$

K.Ono [8] considered the case $\rho(x, u_t) = u_t + |u_t|^r u_t, 0$

$$E(t) \leq C(\|u_0\|_{H_2}, \|u_1\|_{H_1})(1+t)^{\nu-1}$$

with $\nu = \max\{0, (4 - Nr)/2(r+2)\}$. This is refined if $(u_0, u_1) \in H_2 \cap L^p \times H_1 \cap L^p$. To our knowledge, these are the only results for the whole space, though the Klein-Gordon type wave equation with nonlinear dissipations like $|u_t|^r u_t$ have been treated by Nakao [6] and also Mochizuki and Motai [5].

2. Some identities and the basic inequality

Throughout this work we shall use only familiar Sobolev spaces and the definition of them are omitted. Assume Hyp.A. Then , for each $(u_0, u_1) \in H_1^0(\Omega) \times L^2(\Omega)$ the problem (1.1)-(1.3) has a unique solution $u(t)$ in $C_1([0,\infty); L^2(\Omega)) \cap C([0,\infty); H_1^0(\Omega))$, and for each $(u_0, u_1) \in H_1^0(\Omega) \cap H^2(\Omega) \times H_1^0(\Omega)$ the solution belongs to $X_{loc}^2 \equiv W_{loc}^{2,\infty}([0,\infty); L^2(\Omega)) \cap W_{loc}^{1,\infty}([0,\infty); H_1^0(\Omega)) \cap L_{loc}^{\infty}([0,\infty); H_2(\Omega))$, satisfying the estimate

$$\|u_{tt}\| + \|\nabla u_t\| + \|\Delta u(t)\| \leq C(\|\nabla u_1\| + \|\Delta u_0\| + \|\rho(x, u_1)\|) \equiv K_0 < \infty \tag{2.1}$$

for $0 \leq t < \infty$ and some $C > 0$.

These results are standard and well known (cf. Lions and Strauss [3]).

Let us consider the wave equation with a forcing term :

$$u_{tt} - \triangle u + \rho(x, u_t) = f(x,t) \qquad \text{in } \Omega \times [0,\infty) \tag{2.2}$$

$$u(x,0) = u_0, \quad u_t(x,0) = u_1 \qquad \text{in } \Omega \tag{2.3}$$

$$u(x,t) = 0 \qquad \text{on } \partial\Omega \times [0,\infty). \tag{2.4}$$

For a moment we assume that $f \in W^{1,2}_{loc}([0,\infty); L^2(\Omega))$ and $u(t)$ is a solution in X^2_{loc} of the problem (2.2)-(2.4).

For such solutions the following familiar identities hold.(See Lions[2] and Komornik[1]).

Lemma 2.1. *Let $h(x) = (h_1(x), h_2(x), \dots, h_N(x)) \in (W^{1,\infty}(\Omega))^N$ and $\phi(x) \in W^{1,\infty}(\Omega)$. Then,*

$$\frac{d}{dt}E(t) + \int_\Omega \rho(x,u_t)u_t dx = \int_\Omega f u_t dx, \tag{2.5}$$

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \varphi(x)u_t u dx - \int_\Omega \varphi(x)|u_t|^2 dx + \int_\Omega \nabla u \cdot \nabla(\varphi u) dx \\ &+ \int_\Omega \varphi(x)\rho(x,u_t) u dx, \\ &= \int_\Omega f\varphi(x) u dx \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &\frac{d}{dt}\left\{\int_\Omega u_t h(x)\cdot\nabla u dx\right\} + \frac{1}{2}\int_\Omega \nabla\cdot h(x)|u_t|^2 dx \\ &-\frac{1}{2}\int_\Omega \nabla\cdot h(x)|\nabla u|^2 dx + \int_\Omega \sum_{i,j=1}^N \frac{\partial h_i}{\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} dx \\ &-\frac{1}{2}\int_{\Gamma(x_0)} |\frac{\partial u}{\partial \nu}|^2 \nu\cdot h(x) d\Gamma + \int_\Omega \rho(x,u_t) h(x)\cdot\nabla u dx \\ &= \int_\Omega f h(x)\cdot\nabla u dx. \end{aligned} \tag{2.7}$$

We denote by C positive generic constants independent of the initial data and f. $C(s)$ will denote a function of s which depends on s significantly. Our basic inequality reads as follows:

Proposition 2.2. *There exists $T_0 > 0$ such that if $T > T_0$ and $k > 0$ is sufficiently large, then*

$$\begin{aligned} &X(t+T) - X(t) + k\int_t^{t+T}\int_\Omega \rho(x,u_t)u_t dx ds + \varepsilon_1 \int_t^{t+T} E(s) ds \\ \le\ & C\int_t^{t+T}\{\int_\omega |u_t|^2 dx + \int_{\Omega_L} |\rho(x,u_t)|^2 dx + \int_\Omega (|f||u| + |f|^2) dx\} ds \end{aligned} \tag{2.8}$$

with some $C > 0$, and $\varepsilon_1 > 0$, where $X(t)$ is a certain quantity equivalent to $E(t) + \|u(t)\|^2$.

The first term of the right-hand side is the most delicate one to control. But, we can show the following

Lemma 2.3. *There exists $T_0 > 0$, independent of u, such that if $T > T_0$, then the inequality*

$$\int_t^{t+T}\int_{\Omega_L}|u|^2dxdt \leq C_\varepsilon(T)\int_t^{t+T}\{\int_\Omega(|\rho(x,u_t)|^2+|f|^2)dx+\int_{B_L^C\cup\omega}|u_t|^2dx\}ds$$
$$+\varepsilon\int_t^{t+T}E(s)ds \tag{2.9}$$

holds for any fixed number $\varepsilon > 0$.

Now , we set

$$X(t) = \int_\Omega u_t(\phi(r)(x-x_0)-C_0h(x))\cdot\nabla u dx+\int_\Omega(\alpha+\eta^2)u_tudx$$
$$+\frac{1}{2}\int_{\Omega_R^C}\alpha a(x)|u|^2dx+kE(t). \tag{2.10}$$

It remains to show that $X(t)$ is equivalent to $E(t)+||u(t)||^2$. Indeed, we have :

Lemma 2.4. *For a large $k > 0$, there exist constants $C_1 > 0$ and $C_2 > 0$ such that for any $t \geq 0$,*

$$C_1(E(t)+||u(t)||^2) \leq X(t) \leq C_2(E(t)+||u(t)||^2). \tag{2.11}$$

3. Main Results

We shall prove the bound of $\|u(t)\|$ and decay estimate of the energy $E(t) \equiv \frac{1}{2}(\|u_t(t)\|^2+\|\nabla u(t)\|^2)$ as follows:

Theorem. *Let $u(t) \in X_{loc}^2$ be a solution of (1.1)-(1.3) satisfying (2.1). Under Hyp.A we have the estimates*

$$||u(t)||^2 \leq C(Q_0+\|u_0\|^2)(t+1)^\nu \text{ and } E(t) \leq C(Q_0+\|u_0\|^2)(t+1)^{\nu-1},$$

where ν and Q_0 are defined corresponding to the cases as follows:

(1) *If $0 \leq p < \infty$ and $0 \leq q \leq 2/(N-2)^+(0 \leq q < \infty$ if $N = 1,2)$, then*

$$\nu = \max\{p/(p+2), q(N-2)^+/(4-(N-2)^+q)\},$$
$$(\nu = \max\{p/(p+2),\varepsilon\}, 0 < \varepsilon << 1, \text{ if } N = 2)$$

and

$$Q_0 = E(0)^{2/(p+2)}+E(0)+(K_0^2+E(0))^{(q+1)\theta}E(0)^{2(q+1)(1-\theta)/(q+2)},$$

where $\theta = Nq/(q+1)(4-(N-2)^+q)$ $\quad(\theta = (q+2)\varepsilon/(q+1)$ if $N = 2)$.

(2) *If $-1 < p \leq 0$ and $0 \leq q \leq 2/(N-2)^+$, then*

$$\nu = \max\{-p/(p+2), q(N-2)^+/(4-(N-2)^+q)\},$$
$$(\nu = \max\{-p/(p+2),\varepsilon\}, 0 < \varepsilon << 1, \text{ if } N = 2)$$

and

$$Q_0 = E(0)+(K_0^2+E(0))^{(q+1)\theta}E(0)^{2(q+1)(1-\theta)/(q+2)},$$

where $\theta = Nq/(q+1)(4-(N-2)^+q)$ $\quad(\theta = (q+2)\varepsilon/(q+1)$ if $N = 2)$.

(3) *If* $0 \le p < \infty$ *and* $-1 < q \le 0$, *then*

$$\nu = p/(p+2) \quad (\nu = \max\{p/(p+2), \varepsilon\} \text{ if } N = 2)$$

and

$$Q_0 = E(0)^{2/(p+2)} + E(0) + (K_0^2 + E(0))^{(q+1)\theta} E(0)^{4/(4-(N-2)^+ q)}$$

with $\theta = -Nq/(q+1)(4-(N-2)^+ q) \quad (\theta = \varepsilon/(q+1)$ *if* $N = 2)$.

(4) *If* $-1 < p \le 0$ *and* $-1 < q \le 0$, *then*

$$\nu = \max\{-p/(p+2), q(N-2)^+/(4-(N-2)^+ q)\},$$

$$(\nu = \max\{-p/(p+2), \varepsilon\}, 0 < \varepsilon << 1, \text{ if } N = 2)$$

and

$$Q_0 = E(0)^{2/(p+2)} + E(0) + (K_0^2 + E(0))^{(q+1)\theta} E(0)^{4/(4-(N-2)^+ q)}$$

with $\theta = -Nq/(q+1)(4-(N-2)^+ q) \quad (\theta = \varepsilon/(q+1)$ *if* $N = 2)$.

The estimates are valid also for the Cauchy problem, $\Omega = \mathbb{R}^N$, *if* $N \ge 3$, *where the boundary condition* $u|_{\partial\Omega} = 0$ *should be dropped.*

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