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ON REACTION-DIFFUSION EQUATIONS IN POPULATION DYNAMICS

INKYUNG AHN

Abstract. In this survey article, we provide some basic population models and state positive coexistence results for such models. Then certain biological models with self-cross diffusion rates are introduced.

In this expository article, we represent various model from ODE systems to reaction-diffusion equations in population dynamics. First, we begin with the motivation of basic models with constant diffusions and linear growth rates to present the population dynamics. Then, after we discuss the existence results for the systems with the nonlinear growth rates, the system with self-cross diffusions will be introduced from the point of view of steady states. We will focus on the existence of positive solutions to the models.

To describe the interaction between two species or chemicals, one of the earliest models used was the Lotka-Volterra ODE. The following logistic model, which allows for the limited resources and environmental reactions, is such a model.

$$\begin{cases} \dfrac{du}{dt} = u(a - bu \pm cv) \\ \dfrac{dv}{dt} = v(e - fv \pm gv). \end{cases}$$

Here u and v are densities of two interacting species or chemicals and the constants a, b, c, e, f and g are all positive.

In ODE models it is assumed that the population or chemicals are uniformly distributed over the spatial region of interest. However, it is natural to expect a higher or lower concentration of species will occur at different locations due to the availability of water and food. Such concentrations have given rise to reaction-diffusion models, such as

$$(1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + u(a - bu \pm cv) \\ \dfrac{\partial v}{\partial t} = d_2 \Delta + v(e - fv \pm gv), \quad \text{in } \Omega \times \mathbf{R}^+. \end{cases}$$

The positive constants d_1 and d_2 are the rates of diffusion of the two population or chemicals, and Ω is a bounded open region in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$.

A great amount of studies has been achieved relating to the system (1); Some results can be found; Blat and Brown[1], Dancer[3] for predator-prey interaction model; Pao[12], Korman and Leung[4], McKenna and Walter[9], Cosner and

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Lazer[2] for competition model; For symbiotic model, Korman and Leung[4], McKenna and Walter[9].

The following Kolmogorov model allows nonlinear interaction rates.

$$\begin{cases} -d_1\Delta u = uM(u,v) \\ -d_2\Delta v = vN(u,v), \quad \text{in } \Omega. \end{cases} \tag{2}$$

These equations have been used to model the three classical ecological interactions of two living species: predator-prey, competition, and symbiosis, where different modes of interaction are determined by the sign of M_v and N_u as follows:

(i) Predator-prey, $M_v < 0$, $N_u > 0$. Here u is the prey v is the predator.

(ii) Competition, $M_v < 0$, $N_u < 0$.

(iii) Symbiosis, $M_v > 0$, $N_u > 0$.

We also have the additional restrictions of $M_u,\ N_v < 0$, due to the limited resources, i.e., the environment can only support up to a certain number of each species.

We say that the system (2) has a positive solution (u,v) if $u(x) > 0$ and $v(x) > 0$ for all $x \in \Omega$. The existence of a positive solution (u,v) to system (2) is called a *positive coexistence*. Positive coexistence has important implications to the long term behavior of the biological system under investigation. One of the basic questions regarding the interacting system (2) is the following: can the coexistence be predicted using the information on the individual behaviors of the two species? Let us denote by u_0, v_0 the densities of the two species with their partner or rival in absence. Thus u_0, v_0 represent the individual behaviors of u, v before their participation into the interaction. One answer to the above question has been supplied by extensive work in last twenty years: for predation and competition models under homogeneous Dirichlet or Robin boundary conditions, the existence of positive solutions to the system (2) is closely related to the sign of $\lambda_1(\Delta + M(0,v_0))$ and $\lambda_1(\Delta + N(u_0,0))$. The principal eigenvalues of two differential operators obtained from linearizing system (2) at $(u_0,0)$ and at $(0,v_0)$, respectively. These results reflect the fact that the positive coexistence depends on the data u_0, v_0 for a given domain on which the interaction takes place.

In [5], Li gave necessary and sufficient conditions for the existence of positive solutions of steady states for predator-prey system under Dirichlet boundary conditions on Ω in $\mathbf{R}^n$ for $d_1 = 1$, $N(u,v) = g(u) - m(v)$.

Throughout this article, λ_1 denote the principal eigenvalue of $-\Delta$ under homogeneous boundary conditions.

Theorem 1. *If $m \not\equiv 0$, then the positive solution (u,v) has a priori bounds $0 \le u \le B_1$, $0 \le u \le B_2$.*

(i) if $M(0,0) \le \lambda_1$, $g(0) \le \lambda_1 d_2 + m(0)$, then $(0,0)$ is the only nonnegative solution.

(ii) If $g(0) < \lambda_1 d_2 + m(0)$, then the system has a positive solution iff $M(0,0) > \lambda_1$ and the first eigenvalue of the operator $d_2\Delta + [g(u_0) - m(0)]I$ is positive.

(iii) If $g(0) > \lambda_1 d_2 + m(0)$, we assume $M(0,v) \ge M(u,v)$ for u, $v \ge 0$. Then the system has a positive solution iff $M(0,0) > \lambda_1$ and the first eigenvalue of the operator $d_2\Delta + [g(u_0) - m(0)]I$ and $\Delta + M(0,v_0)I$ are both positive.

In [6], Li and Logan give necessary and sufficient conditions for the coexistence of strictly positive solutions of steady states for competing interacting system between two species under Dirichlet boundary condition for $M(u,v) = f(u) - g(v)$ and

$N(u,v) = b(v) - a(u)$. They showed that the coexistence is closely related to the spectral properties of certain differential operators of Schrodinger type.

Theorem 2. *(i) If either* $f(0) \leq \lambda_1$ *or* $b(0) \leq \lambda_1$*, then the system has no strictly positive solutions.*

(ii) If $f(0) > \lambda_1$*,* $b(0) > \lambda_1$ *and the first eigenvalues of the operators* $\Delta + (f(0) - g(v_0))I$ *and* $\Delta + (b(0) - a(u_0))I$ *have the same sign, i.e. if both of them are either positive or negative or zero, then the system has a positive solution.*

(iii) If $f(0) = b(0)$ *and both* $f' + a'$*,* $b' + g'$ *have the same constant sign on* $(0, c)$ *where* $c = \max\{c_0, c_1\}$ *then the conditions in (ii) are necessary and sufficient for the existence of positive solutions to the system*

Biologically, when $f' + a' > 0$ and $b' + g' > 0$, the model describes a highly competitive system, while when $f' + a' < 0$, $b' + g' < 0$, they model a weakly competitive system.

In [7], Li and Ghoreishi established sufficient and necessary conditions for the existence of positive solutions of steady states for symbiotic interaction model under Dirichlet boundary conditions.

The main results is that the existence of positive solutions is largely governed by the status of the constant equilibria of the corresponding ODE system

$$du/dt = uM(u,v), \quad dv/dt = vN(u,v).$$

In comparison, the coexistence of positive solutions to predator-prey or competition interaction models depends on the shape of the domain because such an existence is characterized by the spectral properties of certain differential operators of Schrodinger type.

In recent years there has been a considerable amount of interest to the following model with the linear diffusion and growth rates :

$$\begin{cases} -\Delta[(\alpha_1 + \beta_{11}u + \beta_{12}v)u] = u(a_1 - b_{11}u - b_{12}v) \\ -\Delta[(\alpha_2 + \beta_{21}u + \beta_{22}v)v] = v(a_2 - b_{21}u - b_{22}v) \quad \text{in } \Omega. \end{cases} \tag{3}$$

This system (3) was proposed first by Shigesada *et al.* in [16]. The idea is that the main reason of dispersal of two competing species is population pressures due to the mutual interference between the individuals.

For one dimensional domain, there are several works relating to the existence of non-constant solutions to the systems (3) under homogeneous Neumann boundary conditions. ([10], [11]) They showed that non-constant positive solutions exist when α_2, β_{21}, β_{22} are sufficiently small. In [17], the system (3) was considered under a homogeneous Dirichlet boundary conditions using the singular perturbation. He found the positive solutions if certain parameters are sufficiently small. For n-dimensional domain, W. Ruan[13] considered the coupled competition elliptic system (3) with Dirichlet boundary conditions by using the index theory. In [8], Y. Lou and W. Ni investigated the existence of non-constant solutions of the above system (3) under Neumann boundary conditions employing the method of Lyapunov functional and degree theory. Recently, Ryu and Ahn[15] provided sufficient conditions for the existence of positive solutions to the system (3) under homogeneous Robin boundary conditions. They also studied the positive coexistence to the corresponding system of (3) with predator-prey interactions.

In [14], Ryu and Ahn investigated the existence of positive solutions to the elliptic competing interacting system with self-cross diffusions :

$$(4) \qquad \begin{cases} -\Delta[\varphi(u,v)u] = uf(u,v) \\ -\Delta[\psi(u,v)v] = vg(u,v) \qquad \text{in } \Omega, \end{cases}$$

under Robin boundary conditions. They gave sufficient conditions for the coexistence of positive solutions of system (4) with competitive interactions by using the method of the fixed point index of compact operators in a positive cone. In fact, they showed that if the sign of the principal eigenvalue of operators $\Delta\varphi((0,v_0)+f(0,v_0)$, $\Delta\psi(u_0,0)+g(u_0,0)$ is both positive or both negative or both equal to zero, then the system (4) has a positive solution. Specially, we should point out that when one apply this result to the model (3), there are interesting observations as follows: (i) increasing self-diffusion pressure eliminate all possible nontrivial steady states, (ii) increasing the cross-diffusion pressures tends to create nontrivial steady state and (iii) increasing the diffusion coefficients α_1, α_2 while everything else is fixed, tends to eliminates any existing positive steady states. These implications follow from the variational property of the principal eigenvalue of certain differential operators and *a-priori* estimates of positive solutions to the system (3).

It is known that various models with self-cross diffusions have appeared in many fields of applied sciences. Thus we hope to understand mathematically the dynamics involved in self-cross diffusions so that one is able to model new phenomena more precisely.

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Department of Mathematics, Korea University, Jochiwon, Chung-nam 339-700, Korea
E-mail address: ahnik@korea.ac.kr