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ASYMPTOTIC STUDY OF NAVIER-STOKES FLOWS

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Abstract. This article surveys the asymptotic theory of high Reynolds number steady flows in two dimension. Several research topics are discussed with current achievements of relevant subjects.

1. Introduction

Among the many branches of applied mathematics, mathematical fluid dynamics has been regarded as an excellent model of very successful and fruitful correspondence with real physical experiments. This prominent feature results in a great deal of research outputs throughout many centuries, which are too broad to describe here. In this survey, we concentrate on the asymptotic properties of two dimensional incompressible Navier-Stokes flows for very large Reynolds number. The Reynolds number (Re) which is essentially reciprocal to the physical viscosity is a positive real parameter determining the state of the solution(s) of Navier-Stokes equations. In general, if Re is small, the flow is dominated by the viscous stress and the motion is quite predictable. As Re increases the flow is affected by the advection as well as viscous force simultaneously, which makes the corresponding movement of fluid gradually unstable. After this stage, for larger Re, some steps of branching (or bifurcations) of solution occur and the flow eventually becomes chaotic and turbulent. Consequently, the limit flow of infinite Reynolds number is normally related to turbulence which is a vast, long standing and still active subject.

Turbulence describes a terminal state of infinite Reynolds number flow, and only statistical quantities are considered meaningful. In contrast to this, asymptotic study of Navier-Stokes flows deals with the intermediate process for the flow to reach the (final) turbulent state. Sometimes the theoretical study does not give a satisfactory answer due to the overall instability of high Reynolds number flows, and one needs to control the setting or the configuration of the flow. This is another difficulty in studying high Reynolds number flows. Nevertheless, it is very important to study the asymptotic properties for the following reasons. First of all, we obtain some clues on general properties of the Navier-Stokes equations. Secondly, in spite of certain difference, real physical flows at large Re have enough in common with the asymptotics at large Re, which can be used for qualitative analysis of real turbulent flows.

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2. An Example: Flow past bluff bodies

Historically, Prandtl(1905) first noticed the emerging of boundary layer if a viscous flow pass around an obstacle at large Reynolds number. As Re increases, the thickness of boundary layer from the obstacle decreases and in the limit $Re \to \infty$ (we call this the *inviscid limit*) it becomes zero. In addition, he observed the two opposite eddies after the obstacle for moderately large Re.(Due to the instability, the two eddies are shedding from the obstacle for larger Re. See Van Dyke(1988), Chapter 3.) He also found that the vorticity approaches to uniform constant over each eddy of closed streamline region as $Re \to \infty$. Combining these facts, the possible inviscid limit of a viscous flow, if stability is guaranteed, is consisted of regions of closed streamlines with constant vorticities and some separating discontinuity interfaces which are the limits of boundary layer along the boundary and interior layer. This asymptotic idea was rediscovered independently by Batchelor(1956) and is now called *Prandtl-Batchelor(PB)* theory. Corresponding flows are called *Batchelor* (or *Prandtl-Batchelor*) flows (Caflisch 1988).

This kind of flow is practically important since it provides reasonable answers to numerous physical problems, for example in aerodynamics.(Bunyakin *et al* 1996) Among them, probably most important, is the flow after bluff obstacle(s). The flow after a circular cylinder, for instance, has been at the center of innumerable researches invoking a famous controversy on the inviscid limit(Smith 1985). There has been many theories conjectured to predict the flow as $Re \to \infty$.(We remark that there has been more than 30 distinguished possible limits of the model!) Powerful computing has been helping to select meaningful and correct guesses of various ideas. Despite much effort, it is quite recent (Chernyshenko 1988) that this problem is finally settled down in two dimension. In brief, with the aids of the matched asymptotic expansion, five compound structures in different scales are found as $Re \to \infty$. This answer contains an eddy-scale flow, cyclic (periodic) boundary layer, body scale flow most of all. Details are well reviewed by Chernyshenko(1998). Validity is confirmed by comparisons with direct numerical calculations.

3. Prandtl-Batchelor theory

To continue we shortly describe the outline of PB theory. We shall be concerned with the two-dimensional steady incompressible Navier-Stokes equations for a velocity $\mathbf{u} = \mathbf{u}(\mathbf{x}) = (u, v)$, where $\mathbf{x} = (x, y)$ in R^2:

$$\mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - Re^{-1} \nabla^2 \mathbf{u} = 0, \quad \nabla \cdot \mathbf{u} = 0, \tag{1}$$

where p is the pressure and the positive real parameter Re is the Reynolds number. As $R \to \infty$ the viscous term in (1) is dropped, yielding the Euler equations

$$\mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0, \quad \nabla \cdot \mathbf{u} = 0. \tag{2}$$

Alternatively, we will use the streamfunction ψ defined by $u = \partial\psi/\partial y, v = -\partial\psi/\partial x$. Introducing the vorticity $\omega = \partial v/\partial x - \partial u/\partial y = -\nabla^2\psi$, the curl of (1) yields

$$\mathbf{u} \cdot \nabla \omega - R^{-1} \nabla^2 \omega = 0. \tag{3}$$

For $R \to \infty$ the corresponding Euler equation is $\mathbf{u} \cdot \nabla\omega = \partial(\omega, \psi)/\partial(x, y) = 0$, indicating ω is a function of ψ alone.

Let D be a simply connected bounded domain in two dimension. We assume a solution of (3) with nested closed streamlines where, for $R > R_0 > 0$ the streamlines form a simple family of closed contours. We call a flow of this kind a *vortical eddy* or simply an eddy. Also we suppose that at any point in the interior of D the viscous term is negligible in the limit of large R, and specifically that ω may there be replaced by a function of ψ alone, $\omega(\psi)$. The boundary ∂D is taken to be defined by $\psi = 0$, and $0 < \psi < c$ throughout the interior of the eddy. We may assume $\omega > 0$ in the eddy, in which case c is a positive constant determined by the magnitude of the vorticity. Let $\psi = c_1,\ 0 < c_1 < c$ be a streamline, and consider the domain D_1 within this streamline. Applying Green's theorem to (3) we obtain

$$\oint_{\psi=c_1} (u_n\omega - \frac{1}{R}\frac{\partial\omega}{\partial n})\ \ ds = 0, \tag{4}$$

s is the arclength. Since by definition $u_n = 0$ on a streamline, (4) reduces to

$$\frac{1}{R}\oint_{\psi=c_1} \frac{\partial\omega}{\partial n}\ \ ds = 0. \tag{5}$$

The R^{-1} factor may at this point be discarded. If we then assume that R is so large that ω may be replaced by a function having the Euler form $\omega(\psi)$, (5) reduces to

$$\gamma(c_1)\omega'(c_1) = 0, \tag{6}$$

where $\gamma(\psi)$ is the circulation integral within the eddy,

$$\gamma(c_1) = \oint_{\psi=c_1} q\ \ ds, \quad q^2 = u^2 + v^2. \tag{7}$$

Since c_1 determines an arbitrary streamline within the eddy, and since $\gamma > 0, 0 < \psi < c$ for the eddy, the conclusion is that $\omega = \omega_0 =$ constant for $0 \le \psi < c$.

This is plausible under the conditions stated, but as far as we are aware there has not been a mathematical proof of the result based solely on the form of the domain and the boundary data.(Kim 1998, Okamoto 1994) One can, of course, give *a priori* conditions on ω and ψ sufficient to establish Prandtl's result rigorously. Indeed, sufficient conditions are contained in the following,

Theorem 1. *(Prandtl) Let $(u, v), p$ be a solution of* (1) *defined over D for $R > R_0$. Assume, for any positive number $\delta < c$, (1) the corresponding stream function $\psi(x, y)$ has continuous second derivatives, and that $0 \le \psi \le c(R) \le c_0$ over D, with $\partial D : \psi = 0$, (2) there is a function f depending upon ψ alone such that $|\frac{\partial\omega}{\partial\psi} - f'(\psi)| = o(R) \cdot |\frac{\partial\omega}{\partial\psi}|$, both uniform for $R > R_0$ and $\delta \le \psi$. Then necessarily $f = \omega_0 =$ constant for $\delta \le \psi \le c(\infty)$.*

4. Research interests

Again, there are various aspects of the asymptotics of Navier-Stokes flows and here are some brief skectch of such topics known to the author.

The first mathematical question is validity of Prandtl's theorem. Reasonable explanation can be found as in previous section, however, no rigorous analysis has been done to confirm the constant vorticity state result. Kim(2000) considers a simple case of a slightly perturbed solid body rotation in a circular domain, in which case, the desired Prandtl-Batchelor state is indirectly obtained by analytically proving the existence and uniqueness of the corresponding boundary layer. The basic flow of solid body rotation is absolutely stable, which admits confirming numerical computation(Kim 1998, Kim and Lee 1999). The next step is to consider the correctness of Prandtl-Batchelor theory for more general flows such as flows in some closed convex domains where the pressure is not constant in general. (See Acker 1998, 2002 and Vynnycky 1998.) For example, in the case of an elliptical domain, one needs to adopt more elaborate analytical technics such as multiple scale expansions(Edwards 1997). There is no simple analogous of PB theory in three dimension(Mezic 2002).

Second, the limiting constant vorticity is another concern. In the circular domain, we obtain the Batchelor-Wood formula on the constant value by not so difficult reasoning(Batchelor 1967, Squire 1967). In the process, viscous effect along the boundary should be considered to be matched with the outer nearly inviscid flow. For a general domain with more than one eddy we need some generalized scheme to calculate the constant vorticity of each subeddy. The effect of boundary layer of nonzero pressure gradient should be incorporated and thus make the problem much harder. For the simple case where two opposite eddies are perturbed by certain boundary velocity is considered (Kim and Childress 2001) and the first order term of the asymptotic expansion was deduced. Feynman and Lagerstrom(1967) interestingly claims that the Batchelor-Wood value should be used as a first order approximation for a general domain where the boundary velocity is slightly different with the inviscid limit velocity such that the eddy configuration is reasonable. In the case of the three dimensional spherical drop, similar argument produces another formula(Harper and Moore 1968).

Third, we are interested in the configuration of the flow. In fact, this is our final aim to determine the flow completely. To do so, we first determine the streamline topology under the given boundary condition and sufficiently large Reynolds number. Several or more eddies are possible producing a complicated interfaces which actually are free boundaries. Here, an essential ingredient is the stability of the flow since flows become unstable as Re increases. Thus, under given boundary and boundary velocity we have to determine the free boundaries which make the given flow stable. This might be very difficult Poisson type free boundary problem of which the simplest case is considered by Kim(1999).

Finally, we refer another aspect: numerical methods. To compute the Prandtl-Batchelor flows effectively adaptive resolutions are desirable since it contains a thin boundary layer. Kim and Lee(1999) utilizes the fast Poisson adaptive solver for this purpose and obtained a better result. Acker *et al*(1997) presents another approach to this problem.

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