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UNIQUENESS OF OPTIMAL CONTROLS FOR DISTRIBUTED PARAMETER SYSTEMS

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Abstract. We will survey the technique proving the existence and uniqueness of the optimal controls for the controlled system governed by semilinear parabolic equations.

1. Introduction

We consider a distributed optimal control problem governed by a semilinear parabolic equation, where a constraint on the control is given. In this case the optimal control problem is to seek a control function belonging to an admissible control set so as to minimize an introduced functional. That is, we study the following optimal control problem governed by a semilinear state equation.

(P) Find $u^* \in U_{ad}$ such that

$$J(u^*) = \inf_{u \in U_{ad}} J(u)$$

subject to

$$\frac{dy}{dt} + Ay = F(u, y) \quad \text{in} \quad (0, T), \quad y(0) = y_0,$$

where U_{ad} is an admissible set of control variables and $(0, T)$ is the time interval with $T < \infty$.

For a long time the problem (P) has been studied as the existence problem and characterizing problem of u^*, which is referred to as an optimal control. Thanks to the numerous studies it is not difficult any more to prove the existence of u^* and to describe the necessary conditions on u^*. We refer to [2, 3]. By comparison with these studies, there are a few researches of the uniqueness of u^* and it is not easy to show the uniqueness. Generally a local uniqueness of u^* can be solved if we utilize the inverse function theory or show that the introduced cost functional is strictly convex. But these arguments don't give an explicit estimate for the upper bound of data.

In this lecture we investigate the optimal control problem where the state equation is the semilinear parabolic equation. Specially we will focuss on the uniqueness of the optimal control.

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2. General notations

Let Ω is an open subset of R^N with the smooth boundary Γ, $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$. We introduce the Hilbert spaces V and $H = L^2(\Omega)$ with the norms $\|\cdot\|$ and $|\cdot|$, respectively. By V' denotes the dual space of V with the norm $\|\cdot\|_{V'}$. Let us introduce the Gelfand triple spaces:

$$V \hookrightarrow H \equiv H' \hookrightarrow V',$$

where all the embeddings are continuous and dense. For convenience set $\mathcal{V} = L^2(0, T; V)$, $\mathcal{H} = L^2(0, T; H) = L^2(Q)$ and $\mathcal{V}' = L^2(0, T; V')$ with the norms $\|\cdot\|_{\mathcal{V}}, \|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{V}'}$, respectively.

Let us define the bilinear form $a(\cdot, \cdot)$ on V satisfying

$$|a(\phi, \psi)| \leq \alpha\|\phi\|\ \|\psi\|, \quad \exists \alpha > 0,\ \forall \phi, \psi \in V, \tag{1}$$

$$a(\phi, \phi) + \lambda\|\phi\|^2 \geq \beta\|\phi\|^2, \quad \exists \lambda > 0,\ \beta > 0, \quad \forall \phi \in V. \tag{2}$$

Then we can define the linear operator $A : V \to V'$ satisfying $a(\phi, \psi) =< A\phi, \psi >$, where $< \cdot, \cdot >$ denotes the dual pairing between V' and V. In (2), we take $\lambda = 0$, which is possible. Let us define a Nemytsky operator F on $\mathcal{H}$ by setting

$$F(y)(x, t) = f(x, t, y(x, t)), \quad \forall (x, t) \in Q.$$

2.1. The case of $f(x, t, y(x, t)) = \sum_{i=1}^N u_i(t)\frac{\partial y}{\partial x_i}(x, t)$.

Here we survey the article studied in A. Addou and A. Benbrik [1]. He studied the following minimization problem.

Minimize

$$J(u) = \int_0^T |u(t)|^2_{R^N}\, dt + \int_0^T \|y(t)\|^2_{L^2(\Omega)}dt$$

subject to

$$\frac{\partial y}{\partial t} + \Delta^2 y = \sum_{i=1}^N u_i(t)\frac{\partial y}{\partial x_i} \quad \text{in} \quad Q,$$

$$y = \frac{\partial y}{\partial \nu} = 0 \quad \text{on} \quad \Sigma \quad \text{and} \quad y(x, 0) = y_0(x) \quad \text{in} \quad \Omega.$$

In this case, we take $V = H_0^2(\Omega)$ with the norm $\|\phi\| = a(\phi, \phi)$, which is the equivalent norm defined on $H_0^2(\Omega)$. From this, $V' = H^{-2}(\Omega)$ and $A = \Delta^2$. Since the embedding $V \hookrightarrow H$ is compact, the embedding $\mathcal{W} = \{\phi \in \mathcal{H} : \phi \in \mathcal{V}, \phi' \in \mathcal{V}'\} \hookrightarrow \mathcal{H}$ becomes compact. Let us define $B = (B_1, B_2, \cdots, B_N)$, where $B_i : V \to H, \phi \mapsto \frac{\partial \phi}{\partial x_i}$, is linear and continuous. As the control space, we take $\mathcal{U} = L^2(0, T; R^N)$ and set

$$u(t)By(t) = \sum_{i=1}^N u_i(t)B_i y(t) \in \mathcal{V}', \quad y(t) \in V.$$

Then the partial differential equations are changed by the abstract evolution equations in H:

$$\dot{y}(t) + Ay(t) = u(t)By(t) \quad \text{in} \quad (0, T), \quad y(0) = y_0. \tag{1}$$

Also the cost function becomes

$$J(u) = \|u\|_{\mathcal{U}}^2 + \|y\|_{\mathcal{H}}^2. \tag{2}$$

We state the existence and uniqueness of solutions for (1).

Theorem 2.1. *Assume that $u \in \mathcal{U}, y_0 \in H$ and $f \in \mathcal{V}'$. Then there is a unique solution $y \in \mathcal{W}$ of*

$$\dot{y}(t) + Ay(t) = u(t)By(t) + f(t) \quad in \quad (0, T), \quad y(0) = y_0. \tag{3}$$

Furthermore, the following inequalities hold:

$$\|y\|_{\mathcal{V}} \leq \frac{1}{\sqrt{2}}|y_0| + \|f\|_{\mathcal{V}'}, \tag{4}$$

$$\|y\|_{L^\infty(0,T;H)} \leq |y_0| + \sqrt{2}\|f\|_{\mathcal{V}'}. \tag{5}$$

In the proofs of (4) and (5) the important fact is $< u(t)By(t), y(t) >= 0$. Since the solution y of (1) is uniquely depending on u, we can define the solution map $Y : \mathcal{U} \to \mathcal{W}$ as $Y(u) = y$, which is the solution of (1) for a given $u \in \mathcal{U}$.

The minimizing problem mentioned at the beginning in this section is described as follows:

Minimize $J(u)$ over $\mathcal{U}$ subject to (1) and (2).

Thus is, find u^* such that $J(u^*) = \inf\{J(u) : u \in \mathcal{U}\}$. We refer to such a u^* an optimal control and call $y^* = Y(u^*)$ the corresponding optimal trajectory.

Theorem 2.2. *Assume that $y_0 \in H$. Then there exists an optimal pair $(u^*, y^*) \in \mathcal{U} \times C([0, T]; H)$.*

The fact that the embedding $\mathcal{W} \hookrightarrow \mathcal{H}$ is compact is crucial in the proof of this theorem.

To deduce the necessary conditions on u^* we must prove that the map $Y(u)$ is Gâteaux differentiable at u^*. This is follows from Lemma 2.1.

Lemma 2.1. *The map $Y : \mathcal{U} \to C([0, T]; H)$ is differentiable in the Frechét sense, and $z = Y'(u)h, h \in \mathcal{U}$ satisfies the equation*

$$\dot{z}(t) + Az(t) = u(t)Bz(t) + h(t)BY(u) \quad in \quad (0, T), \quad z(0) = 0. \tag{6}$$

The equation (6) is said to be a transverse equation. We make use of (6) in order to deduce the necessary conditions on u^*.

Theorem 2.3. *An optimal control u^*, the state y^*, and the corresponding adjoint state p^* necessarily satisfy the following equations:*

$$\dot{y}^*(t) + Ay^*(t) = u^*(t)By^*(t) \quad in \quad (0, T),$$

$$-\dot{p}^*(t) + Ap^*(t) = -u^*(t)Bp^*(t) + y^*(t) \quad in \quad (0, T),$$

$$y^*(0) = y_0, \quad p^*(T) = 0,$$

$$u^*(t) = - < p^*(t), By^*(t) >= [- < p^*(t), B_1 y^*(t) >, \cdots, - < p^*(t), B_N y^*(t) >].$$

Theorem 2.4. *If u^* and $\hat{u}$ are the two optimal controls, then*

$$\|u^* - \hat{u}\|_{\mathcal{U}} \leq 3|y_0|^2 \|u^* - \hat{u}\|_{\mathcal{U}}.$$

Therefore if $|y_0| < 1/\sqrt{3}$, then the optimal control is unique.

2.2. **The case of a general nonlinear term.**

Here we will survey the article studied in T. I. Seidman and H. X. Zhou [4]. He considered the controlled system give by

$$\begin{aligned} &\dot{y} + Ay + F(y) = u \ \text{ in } \ Q, \\ &By = 0 \ \text{ in } \ \Sigma \ \text{ and } \ y(0) = 0 \ \text{ in } \ \Omega, \end{aligned} \tag{1}$$

where $u \in \mathcal{H}$, B is a boundary operator suitably related to A.

We state assumptions on A and f. For the linear operator A the solution map, $g \longmapsto y$, of the linear problem

$$\dot{y} + Ay = g \ \text{ in } \ (0,T), \quad y(0) = 0$$

is assumed to be compact provided with $g \in \mathcal{H}$.

For $f : Q \times R \to R$ with $f(\cdot, 0) = 0$ we assume that f satisfies Caratheodory conditions and an one-sided Lipschitz condition

$$-2[f(\cdot, r) - f(\cdot, r')] \le c_1[r - r'] \ \text{ for } \ r \le r'. \tag{2}$$

Then F maps $\mathcal{H}$ into $\mathcal{H}$. Furthermore we assume that $f(x_t, \cdot)$ is differentiable(a.e. $x_t \in Q$) with $G = \partial f / \partial r$ which satisfies

$$|G(\cdot, r) - G(\cdot, r')| \le c_2 |r - r'|. \tag{3}$$

Theorem 2.5. *If f satisfies (2), then the problem (1) has a unique weak solution $y \in \mathcal{W} = \{\phi \in \mathcal{H} : \phi \in \mathcal{V}, \phi' \in \mathcal{V}'\}$. Also y satisfies*

$$\|y\|_{\mathcal{W}} \le c_3 \|u\|_{\mathcal{H}}, \tag{4}$$

where $c_2 = \exp[(c_1 + 1)T/2]$.

Since this solution y is depending on u, we can define the solution map $Y : \mathcal{U} \to \mathcal{W}$ as $Y(u) = y$, which is the solution of (1) for given u. The map Y is continuous from the weak topology of $\mathcal{H}$ to the strong topology of $\mathcal{H}$.

Theorem 2.6. *Let f satisfies (2), (3) and assume*

$$V \subset L^4(\Omega) \ \text{ so } \ \|\phi\|_{L^4(\Omega)} \le c_4 \|\phi\| \ \text{ for } \ \phi \in V. \tag{5}$$

Then the map Y is continuously differentiable from $\mathcal{H}$ to $\mathcal{W}$ with the derivative $Y'(u)h, h \in \mathcal{H}$, which is the solution of

$$\dot{z} + Az + G(y)z = h \ \text{ in } \ (0,T), \quad z(0) = 0.$$

Let us introduce the cost functional as follows:

$$J(u) = \|y - y_d\|^2_{\mathcal{H}} + \|u\|^2_{\mathcal{H}}. \tag{6}$$

Let us consider the minimizing problem described as follows:

$$\boxed{\text{Minimize } J(u) \text{ over } \mathcal{H} \text{ subject to (1) and (6).}}$$

Theorem 2.7. *If f satisfy (2), then there exists an optimal control u^*. If in addition f satisfies (3), then the optimal pair $(u^*, y^*) \in \mathcal{W} \times \mathcal{W}$ satisfies*

$$\begin{aligned} &\dot{y}^* + Ay^* + F(y^*) = u^* \ \text{ in } \ (0,T), \\ &-\dot{u}^* + Au^* + G(y^*)u^* = y_d - y^* \ \text{ in } \ (0,T), \\ &y^*(0) = 0, \quad u^*(T) = 0. \end{aligned}$$

To prove the uniqueness of the optimal control we will give a further restriction on f as follows:

$$-\int G(\phi)\psi^2 dx \leq \theta \|\psi\|^2 \quad \text{for} \quad v, w \in V, \quad \theta < 1. \tag{7}$$

Theorem 2.8. *Suppose that (5) holds and that f satisfies (2), (3) and (7). Then if*

$$J^* = J(u^*) < \rho^2 := \left[\frac{2(1-\theta)}{c_2 c_3 c_4^2}\right]^2, \tag{8}$$

then the optimal control u is unique.

Let us apply the more general minimization problem than (6) subject to (1) as follows:

Minimize $J(u)$ given by

$$J(u) = \|\overline{y} - \overline{y}_d\|_{\mathcal{H}}^2 + \|\overline{u} - \overline{u}_d\|_{\mathcal{H}}^2 \tag{9}$$

subject to

$$\begin{array}{l} \dot{\overline{y}} + A\overline{y} + F(\overline{y}) = \overline{u} \quad \text{in} \quad Q, \\ B\overline{y} = \overline{y}_B \quad \text{in} \quad \Sigma \quad \text{and} \quad \overline{y}(0) = \overline{y}_0 \quad \text{in} \quad \Omega. \end{array} \tag{10}$$

Let y_L be the solution of the linear problem of the form

$$\begin{array}{l} \dot{y}_L + Ay_L = \overline{u}_d \quad \text{in} \quad Q, \\ By_L = \overline{y}_B \quad \text{in} \quad \Sigma \quad \text{and} \quad y_L(0) = \overline{y}_0 \quad \text{in} \quad \Omega. \end{array}$$

If we set $y = \overline{y} - y_L$, then (9) and (10) reduce to

$$J(u) = \|y - y_d\|_{\mathcal{H}}^2 + \|u\|_{\mathcal{H}}^2$$

and

$$\begin{array}{l} \dot{y} + Ay + \tilde{F}(y) = F(y_L) + u \quad \text{in} \quad Q, \\ By = 0 \quad \text{in} \quad \Sigma \quad \text{and} \quad y(0) = 0 \quad \text{in} \quad \Omega, \end{array}$$

where $u = \overline{u} - \overline{u}_d$, $y_d = \overline{y}_d - y_L$ and $\tilde{F}(y) = F(\overline{y}) - F(y_L)$.

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