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AN EPIDEMIC MODEL WITH EXTERNAL FORCE OF INFECTION

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Abstract. In this article we review some recent results for an S-I-R type epdemic model which has extenal force as a source of infection. Our main concern is the existence, uniqueness and stability change for the nontrivial steady states of the model. We also briefly review an S-I-R model without extenal force at the beginning of the article.

0. Introduction

We consider the following system of integro-differential initial-boundary value problem.

$$
\begin{cases}
\dfrac{\partial s}{\partial t}+\dfrac{\partial s}{\partial a}+\mu(a)s=-\lambda(a,t)s,\\
\dfrac{\partial i}{\partial t}+\dfrac{\partial i}{\partial a}+\mu(a)i=\lambda(a,t)s-\gamma(a)i,\\
\dfrac{\partial r}{\partial t}+\dfrac{\partial r}{\partial a}+\mu(a)r=\gamma(a)i,\\
s(0,t)=\displaystyle\int_0^{a_\dagger}\beta(a)\big(s(a,t)+r(a,t)+(1-q)i(a,t)\big)\,da,\\
i(0,t)=q\displaystyle\int_0^{a_\dagger}\beta(a)i(a,t)\,da,\\
r(0,t)=0,\\
i(a,0)=i_0(a),\quad s(a,0)=s_0(a),\quad r(a,0)=r_0(a).
\end{cases}
\tag{0.1}
$$

Here a is the age of individuals, and t is the time. Also, $s(a,t)$, $i(a,t)$ and $r(a,t)$, respectively, denotes the age-specific density of susceptible, infected, and removed individuals.

$\beta(a)$ is the birth rate and $\mu(a)$ is the death rate of the population. $q\in[0,1]$ is the vertical transmission parameter, i.e. the probability that the disease be transmitted from parent to newborn, $\gamma(\cdot)$ is the removal rate of infected individuals, and $\lambda(a,t)$ is the force of infection. Note that since we assume the condition $r(0,t)=0$, our model assumes that there is no vertical transmission of immunity.

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In studying the propagation of a disease through a population, it is frequently assumed that the population is structured into three disjoint classes according to disease status: susceptibles (those who are not infected with the disease but may become infected later), infecteds , and removeds (those who are removed from the dynamics of the disease transmission either through immune or through death caused by the disease). Based on these three classes, a frequently used family of models for disease propagation is called S-I-R [1, 4, 5, 6, 7, 8, 9, 12, 13] (In *S-I-S model*, recovery always takes place without immunity [2, 3, 11], while in *S-I model*, recovery does not take place at all [10]).

Our main concern is the existence, uniqueness and stability change of *endemic* states of the model. An endemic state is a steady state solution of the model for which the density of infected individuals does not vanish identically [4, 6].

Summing the equations in (0.1) we obtain the following problem for the population density $p(a,t) = s(a,t) + i(a,t) + r(a,t)$,

$$\begin{cases} \dfrac{\partial p}{\partial t} + \dfrac{\partial p}{\partial a} + \mu(a)p = 0, \\ p(0,t) = \displaystyle\int_0^{a_\dagger} \beta(a)p(a,t)da, \\ p(a,0) = p_0(a). \end{cases} \tag{0.2}$$

This is the standard McKendrick-Von Foester equation.

We make the following usual hypotheses for this problem,

$$\beta(\cdot) \in L^\infty\big([0,a_\dagger)\big), \qquad \beta(a) \geq 0 \text{ in } [0,a_\dagger), \tag{0.3}$$

$$\mu(\cdot) \in L^1_{\text{loc}}\big([0,a_\dagger)\big), \qquad \mu(a) \geq 0 \text{ in } [0,a_\dagger), \tag{0.4}$$

$$\int_0^{a_\dagger} \mu(a)d\sigma = \infty. \tag{0.5}$$

Here $a_\dagger$ is the maximum age an individual of the population may reach and it may be either finite or infinite. If $a_\dagger = \infty$, we also assume that

$$\textit{there exists } A > 0 \textit{ such that } \beta(a) = 0 \textit{ for } a \geq A. \tag{0.6}$$

Furthermore, in order to deal with a steady state population, we assume that the net reproductive rate of the population is equal to 1 and that the total population is at an equilibrium. This means that

$$\int_0^{a_\dagger} \beta(a)e^{-\int_0^a \mu(\sigma)d\sigma}da = 1, \qquad p(a,t) = p_\infty(a) = b_0\pi(a), \text{ where} \tag{0.7}$$

$$\pi(a) = e^{-\int_0^a \mu(\sigma)d\sigma}. \tag{0.8}$$

Note that the function $\pi(a)$ is the probability that an individual at age 0 can survive until age a. Since no individual may live past age $a_\dagger$, (0.5) is needed.

Clearly, we have to take initial data such that

$$s_0(a) \geq 0, \quad i_0(a) \geq 0, \quad r_0(a) \geq 0, \quad s_0(a) + i_0(a) + r_0(a) = p_\infty(a), \tag{0.9}$$

which gives

$$b_0 = \frac{\int_0^{a_\dagger} s_0(a)da + \int_0^{a_\dagger} i_0(a)da + \int_0^{a_\dagger} r_0(a)da}{\int_0^{a_\dagger} \pi(a)da}. \tag{0.10}$$

We consider the following form for the force of infection:

$$\lambda(a,t) = \kappa(a) \int_0^{a_\dagger} h(\sigma) i(\sigma,t)\, d\sigma + g(a), \tag{0.11}$$

where h is the age-specific infectiousness, κ the age-specific contagion rate, g the external force of infection.

The functions γ, h, κ and g are all nonnegative and in $L^\infty\big([0,a_\dagger)\big)$. We also assume that none of β, μ, γ, h, κ is identically zero.

If g is identically zero, the model does not have any external source of infection [6, 7, 12], while if g is not identically zero, the model does have an external source of infection [4, 5].

1. An S-I-R model without external force

In [6], several strong threshold results are obtained which may be summarized by saying that when a threshold parameter is less than or equal to 1 no endemic state may exist and, when that threshold parameter is larger than 1, then a unique endemic state does exist. The threshold parameter is explicitly computable as

$$T = D(0) + \frac{G\,C(0)}{1-R},$$

where

$$G = \int_0^{a_\dagger} h(a) p_\infty(a) \Gamma(a)\, da, \tag{1.1}$$

$$R = \frac{q}{b_0} \int_0^{a_\dagger} \beta(a)\, p_\infty(a)\, \Gamma(a)\, da, \tag{1.2}$$

$$D(J) = \int_0^{a_\dagger} h(a)\, p_\infty(a)\, F(a,J)\, da, \tag{1.3}$$

$$C(J) = \frac{q}{b_0} \int_0^{a_\dagger} \beta(a)\, p_\infty(a)\, F(a,J)\, da, \tag{1.4}$$

and

$$\Gamma(a) = e^{-\int_0^a \gamma(\sigma)d\sigma}, \tag{1.5}$$

$$F(a,J) = \int_0^a \kappa(\sigma) e^{-\int_\sigma^a \gamma(s)ds - J\int_0^\sigma \kappa(s)ds} d\sigma. \tag{1.6}$$

Several uniqueness results are found there as special cases, but general results concerning uniqueness of endemic states are still open.

Several stability results regarding this model are shown in [7]: They used a mild assumption and proved that the trivial equilibrium is locally asymptotically stable if a threshold parameter is less than 1. Also, when a unique endemic equilibrium exists, the trivial equilibrium is always unstable. The stability of the endemic equilibrium in that case depends on the integral of the removal rate and on the value of that equilibrium at the maximal age. Since both the trivial and the endemic equilibrium may be unstable at the same time, it is quite likely that the model may admit periodic solutions in such case. Some related results were considered in [1, 12, 13] for the same model with constant removal rate γ and no vertical transmission, $q = 0$.

In general, we still do not know whether the sufficient condition they used in order to prove the local asymptotic stability of endemic equilibria is also necessary.

2. S-I-R model with external force of infection – existence and uniqueness

Here we assume that $g(\cdot)$ is not identically zero and review the existence and uniqueness results which are given in [4].

We will be concerned with the following reduced system derived from (0.1) and (0.7):

$$
\begin{cases}
\dfrac{\partial s}{\partial t}(a,t) + \dfrac{\partial s}{\partial a}(a,t) + \mu(a)s(a,t) = -\lambda(a,t)s(a,t), \\
\dfrac{\partial i}{\partial t}(a,t) + \dfrac{\partial i}{\partial a}(a,t) + \mu(a)i(a,t) = \lambda(a,t)s(a,t) - \gamma(a)i(a,t), \\
s(0,t) = b_0 - i(0,t), \\
i(0,t) = q\displaystyle\int_0^{a_\dagger} \beta(a)i(a,t)da, \\
i(a,0) = i_0(a), \quad s(a,0) = s_0(a).
\end{cases}
\tag{2.1}
$$

Now, we consider the following problem, which is obtained by removing the variable t in (2.1):

$$
\begin{cases}
\text{i)} & \dfrac{\partial s}{\partial a} + \mu(a)s(a) = -\big(J\kappa(a) + g(a)\big)\, s(a), \\
\text{ii)} & \dfrac{\partial i}{\partial a} + \mu(a)i(a) = \big(J\kappa(a) + g(a)\big)\, s(a) - \gamma(a)\, i(a), \\
\text{iii)} & J = \displaystyle\int_0^{a_\dagger} h(a)\, i(a)\, da, \\
\text{iv)} & s(0) = b_0 - i(0), \\
\text{v)} & i(0) = q\displaystyle\int_0^{a_\dagger} \beta(a)\, i(a)\, da.
\end{cases}
\tag{2.2}
$$

It is easy to see that the problem admits the disease-free equilibrium $s^*(a) = p_\infty(a)$, $i^*(a) \equiv 0$, if and only if $g(a) \equiv 0$. Since we are assuming that g is not identically

zero, we have to concentrate on the search of endemic states, that is nonnegative solutions for which $i^*(a)$ does not vanish identically.

In order to investigate the existence of such solutions, we modify problem (2.2) by taking the following new variables, the age profiles respectively of infecteds and susceptibles:

$$u(a) = \frac{i(a)}{p_\infty(a)}; \qquad v(a) = \frac{s(a)}{p_\infty(a)}. \tag{2.3}$$

With these definitions, problem (2.2) becomes

$$\begin{cases} \text{i)} & \dfrac{dv}{da} = -\big(J\kappa(a) + g(a)\big)\, v(a), \\ \text{ii)} & \dfrac{du}{da} = \big(J\kappa(a) + g(a)\big)\, v(a) - \gamma(a)\, u(a), \\ \text{iii)} & J = b_0 \displaystyle\int_0^{a_\dagger} h(\sigma)\, \pi(\sigma)\, u(\sigma)\, d\sigma, \\ \text{iv)} & v(0) = 1 - X, \\ \text{v)} & X = q \displaystyle\int_0^{a_\dagger} \beta(\sigma)\, \pi(\sigma)\, u(\sigma)\, d\sigma. \end{cases} \tag{2.4}$$

Note that integration of (2.4.i) gives

$$v(a) = (1 - X) e^{-\int_0^a (J\kappa(\sigma) + g(\sigma)) d\sigma}. \tag{2.5}$$

Then, substituting (2.5) into (2.4.ii) and integrating the equation we have

$$u(a) = X e^{-\int_0^a \gamma(\sigma) d\sigma} + (1 - X) \int_0^a \big(J\kappa(\sigma) + g(\sigma)\big) e^{-\int_\sigma^a \gamma(s) ds - \int_0^\sigma (J\kappa(s) + g(s)) ds} d\sigma. \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.4.iii) and (2.4.v) we get the following relations,

$$\begin{cases} J = XG + \big(1 - X\big)\big(JM(J) + D(J)\big), \\ X = XR + \big(1 - X\big)\big(JL(J) + C(J)\big), \end{cases} \tag{2.7}$$

where we have introduced the following notation:

$$G = b_0 \int_0^{a_\dagger} h(a)\pi(a)\Gamma(a)\, da, \tag{2.8}$$

$$R = q \int_0^{a_\dagger} \beta(a)\pi(a)\, \Gamma(a)\, da, \tag{2.9}$$

$$M(J) = b_0 \int_0^{a_\dagger} h(a)\, \pi(a)\, F(a, J)\, da, \tag{2.10}$$

$$L(J) = q \int_0^{a_\dagger} \beta(a)\, \pi(a)\, F(a, J)\, da, \tag{2.11}$$

$$D(J) = b_0 \int_0^{a_\dagger} h(a)\, \pi(a)\, H(a, J)\, da, \tag{2.12}$$

$$C(J) = q \int_0^{a_\dagger} \beta(a)\, \pi(a)\, H(a, J)\, da, \tag{2.13}$$

and

$$\Gamma(a) = e^{-\int_0^a \gamma(\sigma)d\sigma}, \tag{2.14}$$

$$F(a,J) = \int_0^a \kappa(\sigma) e^{-\int_\sigma^a \gamma(s)ds - \int_0^\sigma (J\kappa(s)+g(s))ds} d\sigma, \tag{2.15}$$

$$H(a,J) = \int_0^a g(\sigma) e^{-\int_\sigma^a \gamma(s)ds - \int_0^\sigma (J\kappa(s)+g(s))ds} d\sigma. \tag{2.16}$$

We seek solutions of (2.7) such that $J \geq 0$ and $0 \leq X \leq 1$. (Note that $X = u(0)$ and $0 \leq u(0) \leq 1$.) In fact, any such a pair (X^*, J^*) provides a nonnegative solution of (2.4) via (2.5) and (2.6).

Note that if the following conditions are satisfied, then (2.7) reduces to a single equation with two unknowns J and X:

$$R = 1 \quad \text{and} \quad L(J) = C(J) = 0 \quad \text{for all } J. \tag{2.17}$$

Hence in this case (2.7) has continuum of solutions (see [4]). In order to rule out such a pathological case, the following assumption is required in the rest of the article.

(2.18) *All the relations in (2.17) do not hold simultaneously.*

Also note that if we have an endemic equilibrium (J^*, X^*) with $J^* = 0$, we can immediately show that $X^* = 0$ and $C(\cdot) \equiv D(\cdot) \equiv 0$ by (2.4 iii), (2.6), (2.12) and (2.13). Since this situation is rather special, we also rule out such a case in this article. Actually we shall assume the following in the rest of the article:

(2.19) $D(\cdot)$ *is not identically zero.*

Note that the above assumption is equivalent to the following:

$$a_h^+ > a_g^-, \text{ where}$$
$$a_h^+ = \mathrm{Inf}\{A : h(a) = 0 \text{ a.e. in } [A, a_\dagger)\},$$
$$a_g^- = \mathrm{Sup}\{A : g(a) = 0 \text{ a.e. in } [0, A]\}.$$

Hence, in biological point of view, assumption (2.19) is not so restrictive.

Under the assumptions (2.18) and (2.19) we can further reduce the system (2.7) to a single equation. In fact, solving the second equation for X we obtain

$$X = \frac{JL(J) + C(J)}{1 - R + JL(J) + C(J)}, \tag{2.20}$$

which, when substituted into the other equation yields:

$$\big(1 - R\big)\big(J - JM(J) - D(J)\big) + \big(J - G\big)\big(JL(J) + C(J)\big) = 0. \tag{2.21}$$

Thus we consider the continuous function

$$\phi(J) = \big(1 - R\big)\big(J - JM(J) - D(J)\big) + \big(J - G\big)\big(JL(J) + C(J)\big) \tag{2.22}$$

and we analyze its behavior in the interval $(0, \infty)$. The following existence and uniqueness results are given in [4].

Theorem 2.1. *An endemic state always exists.*

Theorem 2.2. *If $q = 0$, then the endemic state is unique.*

Note that the assumption, $q = 0$, in theorem 2.2 implies that there is no vertical transmission of the disease.

3. S-I-R model with external force of infection – stability analysis

In this section we shall consider the local stability of the model.

First we take the following new variables, respectively called the age profiles of infecteds and susceptibles:

$$u(a,t) = \frac{i(a,t)}{p_\infty(a)},$$
$$v(a,t) = \frac{s(a,t)}{p_\infty(a)}.$$

Then (2.1) reduces to the following system:

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial a} = -\kappa(a)J(t)v(a,t) - g(a)v(a,t), \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial a} = \kappa(a)J(t)v(a,t) + g(a)v(a,t) - \gamma(a)u(a,t), \\ v(0,t) = 1 - u(0,t), \\ u(0,t) = q\displaystyle\int_0^{a_\dagger} \beta(a)u(a,t)\pi(a)da, \\ u(a,0) = u_0(a) \quad , \\ v(a,0) = v_0(a) \end{cases} \tag{3.1}$$

with

$$J(t) = b_0 \int_0^{a_\dagger} h(\sigma)u(\sigma,t)\pi(\sigma)d\sigma.$$

Integrating v along the characteristics in (3.1), we get

$$v(a,t) = \begin{cases} v_0(a-t)e^{-\int_0^t [\kappa(\sigma+a-t)J(\sigma)+g(\sigma+a-t)]d\sigma} \text{ for } a \geq t, \\ Y(t-a)e^{-\int_0^a [\kappa(\sigma)J(\sigma+t-a)+g(\sigma)]d\sigma} \text{ for } t \geq a, \end{cases} \tag{3.2}$$

where

$$Y(t) = v(0,t).$$

Using (3.2) and integrating u along the characteristics in (3.1), we have

$$u(a,t) = \begin{cases} e^{-\int_0^t \gamma(s+a-t)ds}\Big[u_0(a-t) + v_0(a-t)\times \\ \qquad \displaystyle\int_0^t [\kappa(\sigma+a-t)J(\sigma) + g(\sigma+a-t)]\times \\ \qquad\qquad e^{\int_0^\sigma [\gamma(s+a-t)-\kappa(s+a-t)J(s)-g(\sigma+a-t)]ds}d\sigma\Big], \text{ for } a \geq t, \\ e^{-\int_0^a \gamma(\sigma)d\sigma}\Big[X(t-a) + Y(t-a)\times \\ \qquad \displaystyle\int_0^a [\kappa(\sigma)J(\sigma+t-a) + g(\sigma)]\times \\ \qquad\qquad e^{\int_0^\sigma [\gamma(s)-\kappa(s)J(s+t-a)-g(s)]ds}d\sigma\Big], \text{ for } t \geq a, \end{cases} \tag{3.3}$$

where

$$X(t) = u(0,t).$$

Using this relation, together with

$$X(t) = q\int_0^{a_\dagger} \beta(a)u(a,t)\pi(a)da,$$

we obtain the following expressions for $t \geq a_\dagger$:

$$\begin{aligned} X(t) = q\int_0^{a_\dagger} \beta(a)\pi(a)\Gamma(a)\Big[X(t-a) + (1-X(t-a))\times \\ \int_0^a \{\kappa(\sigma)J(\sigma+t-a) + g(\sigma)\}\times \\ e^{\int_0^\sigma [\gamma(s)-\kappa(s)J(s+t-a)-g(s)]ds}d\sigma\Big]da, \end{aligned} \tag{3.4}$$

$$\begin{aligned} J(t) = b_0\int_0^{a_\dagger} h(a)\pi(a)\Gamma(a)\Big[X(t-a) + (1-X(t-a))\times \\ \int_0^a \{\kappa(\sigma)J(\sigma+t-a) + g(\sigma)\}\times \\ e^{\int_0^\sigma [\gamma(s)-\kappa(s)J(s+t-a)-g(s)]ds}d\sigma\Big]da. \end{aligned} \tag{3.5}$$

In order to linearize (3.4) and (3.5), we let

$$X(t) = X^* + x(t), \qquad J(t) = J^* + j(t), \tag{3.6}$$

where X^* and J^* satisfy (2.7), (2.20) and (2.21).

After a long calculation, we get the following expressions:

$$\begin{cases} x(t) = \displaystyle\int_0^t Q_1(a)x(t-a)da + \int_0^t Q_2(a)j(t-a)da, \\ j(t) = \displaystyle\int_0^t Q_3(a)x(t-a)da + \int_0^t Q_4(a)j(t-a)da, \end{cases} \tag{3.7}$$

where the convolution kernels Q_1, Q_2, Q_3 and Q_4 are given by

$$\begin{aligned} Q_1(a) &= w(a)\left[1 - \int_0^a N(\sigma,J^*)E(\sigma,J^*)d\sigma\right], \\ Q_2(a) &= \int_a^{a_\dagger} w(s)(1-X^*)\kappa(s-a)\left[E(s-a,J^*) - \int_{s-a}^s N(\sigma,J^*)E(\sigma,J^*)d\sigma\right]ds, \\ Q_3(a) &= \tilde{w}(a)\left[1 - \int_0^a N(\sigma,J^*)E(\sigma,J^*)d\sigma\right], \\ Q_4(a) &= \int_a^{a_\dagger} \tilde{w}(s)(1-X^*)\kappa(s-a)\left[E(s-a,J^*) - \int_{s-a}^s N(\sigma,J^*)E(\sigma,J^*)d\sigma\right]ds, \end{aligned}$$

with

$$\begin{aligned} N(\sigma, J) &= J\kappa(\sigma) + g(\sigma), \\ E(a, J) &= e^{\int_0^a [\gamma(s) - N(s,J)]ds}, \\ w(a) &= q\,\beta(a)\pi(a)\Gamma(a), \\ \tilde{w}(a) &= b_0\,h(a)\pi(a)\Gamma(a). \end{aligned}$$

Taking Laplace transforms in (3.7), we obtain the following characteristic equation (see [11]):

$$Q(\lambda) = \left(1 - \hat{Q}_1(\lambda)\right)\left(1 - \hat{Q}_4(\lambda)\right) - \hat{Q}_2(\lambda)\hat{Q}_3(\lambda) = 0, \tag{3.8}$$

where

$$\hat{Q}_i(\lambda) = \int_0^\infty e^{-\lambda a} Q_i(a)da, \quad i = 1, 2, 3, 4,$$

represents the Laplace transform of $Q_i(a)$.

Our problem is to investigate the stability of steady state solutions of (3.1) through the analysis of (3.8). We shall use the following general result.

If all the roots of (3.8) have negative real part, then the steady state corresponding to (X^*, J^*) is locally asymptotically stable. If there are roots with positive real part, then it is unstable.

In order to obtain the local stability results, we shall need the following assumption [7, 12]:

$$u^*(a_\dagger) < e^{-\int_0^{a_\dagger} \gamma(\sigma)d\sigma}. \tag{3.9}$$

Under the assumption (3.9), we can establish the following two lemmas.

Lemma 3.1. *(3.9) is equivalent to each of the following.*

$$\int_0^{a_\dagger} \gamma(\sigma) e^{\int_\sigma^{a_\dagger} (N(s,J^*) - \gamma(s))ds} d\sigma < 1, \tag{3.10}$$

$$\int_0^{a_\dagger} N(\sigma, J^*) e^{\int_0^\sigma (\gamma(s) - N(s,J^*))ds} d\sigma < 1, \tag{3.11}$$

$$\int_0^{s} \gamma(\sigma) e^{\int_\sigma^{s} (N(\tau,J^*) - \gamma(\tau))d\tau} d\sigma < 1, {}^\forall s \in [0, a_\dagger]. \tag{3.12}$$

Lemma 3.2. *Assume that (3.9) holds. Then the functions $Q_1(\cdot)$, $Q_2(\cdot)$, $Q_3(\cdot)$, $Q_4(\cdot)$ are all nonnegative and $Q(0) > 0$.*

Now we have the stability result:

Theorem 3.3. *Assume that (3.9) holds. Then any endemic equilibrium* (X^*, J^*) *is locally asymptotically stable.*

Proof. Note that $0 \le \int_0^\infty Q_1(a)\,da \le R \le 1$. Hence for any λ with $\Re\lambda \ge 0$, lemma 3.2 implies that

$$\begin{aligned}
|Q(\lambda)| &\ge |1-\hat{Q}_1(\lambda)||1-\hat{Q}_4(\lambda)| - |\hat{Q}_2(\lambda)||\hat{Q}_3(\lambda)| \\
&\ge (1-|\hat{Q}_1(\lambda)|)(1-|\hat{Q}_4(\lambda)|) - |\hat{Q}_2(\lambda)||\hat{Q}_3(\lambda)| \\
&\ge \left(1-\int_0^\infty Q_1(a)da\right)\left(1-\int_0^\infty Q_4(a)da\right) - \int_0^\infty Q_2(a)da \int_0^\infty Q_3(a)da \\
&= Q(0) \\
&> 0.
\end{aligned}$$

Thus there is no root λ of the characteristic equation having $\Re\lambda \ge 0$. □

4. Concluding remarks

We have reviewed age-structured epidemic models of S-I-R type with external force of infection.

If the external force exists, an endemic state always exists. We also have uniqueness reslut for the case where there is no vertical transmission of the disease($q = 0$). This uniqueness result applies to many disease, including most sexually transmitted diseases. General results concerning uniqueness of endemic states are still open.

We have shown that the endemic states are locally asymptotically stable for almost all cases. Note that the existence of external force makes the steady state stable. We have used (3.9) to get the stability result in our model. In general, we do not know whether the sufficient condition we have used in order to prove the local asymptotic stability of endemic equilibria is also necessary. General results concerning stability of endemic equilibria are still open.

As a future work, complete uniqueness and stability analysis might be a reasonable choice.

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