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CONTROLLABILITY FOR SOBOLEV TYPE DIFFERENTIAL EQUATIONS

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ABSTRACT. We treat an abstract Sobolev type control system and study the controllability for its trajectories. Assuming the resovent operator is a compact and using the Schauder fixed point theorem, our result is obtained. Furthermore, an example is discussed.

1. INTRODUCTION

Let X be a Banach space with norm $\|\cdot\|$ and let Y be a Banach space with norm $|\cdot|$. Consider the Cauchy problem for the functional differential equation

$$(Bx(t))' + Ax(t) = Lu(t) + f(t, x(t)), \quad x(0) = x_0, \quad t \in J = [0, a], a > 0 \tag{1.1}$$

where the state $x(\cdot)$ takes values in the Banach space X and control function $u(\cdot)$ is given in $L^2(J, U)$, a Banach space of admissible control functions, with U a Banach space. L is a bounded linear operator from U into Y and the nonlinear operator $f \in C(J \times X, Y)$. The problem of controllability of linear and nonlinear systems represented by ordinary differential equations in finite demensional spaces has been extensively studied. Chuckwu and Lenhart[3] have extended the concept to infinite dimensional systems in Banach spaces with bounded operators. Kwun et al.[5] investigated the controllability and approximate controllability of delay Volterra systems by using a fixed point theorem. Also, Balachandran et al.[1] studied the controllability of nonlinear integrodifferential systems in Banach spaces. For the equation of this Sobolev type, Brill [2] was studied for abstract Cauchy problem for the semilinear evolution equation when $L = 0$. Also, Kartsatos and Parrott [4] have dealt pseudoparabolic problems with operator $A(t, u_t)$ and $L = 0$. In particular, our motivation come from the resently artcile about controllability by Balachandran, Dauer and Balasubramaniam [1]. The paper is organized as follows. In section 2 we give defition, notation and hypotheses for our result. In section 3 we give the proof to main theorem. In section 4 we discuss an example to illustrate the result and such example appears in a variety of phyical problems, for example, in thermodynamics and in the flow of fluid through fissured rocks.

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2. Prelimnaries

The operators $A : D(A) \subset X \to Y$ and $B : D(B) \subset X \to Y$ satisfy the following hypotheses:

(C_1): A and B are closed linear operators.
(C_2): $D(B) \subset D(A)$ and B is bijective.
(C_3): $B^{-1} : Y \to D(B)$ is continuous.

The hypotheses $(C_1), (C_2)$ and the closed graph theorem imply the boundedness of the linear operator $AB^{-1} : Y \to Y$.

(C_4): For each $t \in [0, a]$ and for some $\lambda \in \rho(-AB^{-1})$, the resolvent set of $-AB^{-1}$, the resolvent $R(\lambda, -AB^{-1})$ is a compact operator.

Lemma 2.1. *[6] Let $S(t)$ be a uniform continuous semigroup. If the resolvent set $R(\lambda : A)$ of A is compact for every $\lambda \in \rho(A)$, then $S(t)$ is a compact semigroup.*

From the above fact, $-AB^{-1}$ generates a compact semigroup $T(t)$, $t \geq 0$ and so $\max_{t\in J} \|T(t)\|$ is finite. We denote $M = \max_{t\in J} \|T(t)\|$.

Definition 2.1. *The system (1.1) is said to be controllable on the interval J, if, for every $x_0, x_1 \in X$, there exists a control $u \in L^2(J, U)$ such that the solution $x(\cdot)$ of (1.1) satisfies $x(a) = x_1$.*

(C_5): The linear operator W from U into X, defined by

$$Wu = \int_0^a B^{-1}T(a - s)Lu(s)ds,$$

and there exists a bounded invertible operator W^{-1} defined on $L^2(J, U)/\ker W$ and L is a bounded linear operator.

(C_6): The function f satisfies

(i): For each $t \in J$, the function $f(t, \cdot) : X \to Y$ is continuous, and for each $x \in X$ the function $f(\cdot, x) : J \to Y$ is strongly measurable.

(ii): For each natural number k, there is a function $g_k \in L^1(J)$ such that

$$\sup_{|x|\leq k} |f(t, x)| \leq g_k(t)$$

$$\lim_{k\to\infty} \inf \frac{1}{k} \int_0^a g_k(s)ds = \alpha < \infty.$$

where α is a real number.

Now, we shall study the following integral equation

$$x(t) = B^{-1}T(t)Bx_0 + \int_0^t B^{-1}T(t - s)f(s, x(s))ds + \int_0^t B^{-1}T(t - s)Lu(s)ds \tag{2.1}$$

In the following section, we use the Schauder's fixed point theorem in order to establish controllability theorem for Eq.(1.1) under above hypotheses.

3. Controllability

Theorem 3.1. *If the assumptions $(C_1) \sim (C_6)$ are satisfied, then the system (1.1) is controllable on J provided that*

$$\alpha M\|B^{-1}\|(1 + aM\|L\|\|W^{-1}\|\|B^{-1}\|) < 1.$$

4. Application

We wish to illustrate the controllability result of section 3 by showing their applicability to a partial differential equation with a nonlinear function satisfying Caratheodory conditions.

Consider the following differential equation with control term:

$$\begin{aligned}
&\frac{\partial}{\partial t}(z(t,x) - z_{xx}(t,x)) - z_{xx}(t,x) = Lu(t) + g(t, z_{xx}(t,x)),\\
&z(t,0) = z(t,\pi) = 0,\\
&z(0,x) = z_0(x), \quad x \in [0,\pi], t \in J
\end{aligned} \tag{4.1}$$

We assume that

(A_1) **:** The operator $L : U \to Y$, with $U \subset J$, is a bounded linear operator.

(A_2) **:** The linear operator $W : U \to X$ is defined by

$$Wu = \int_0^a B^{-1}T(a-s)Lu(s)ds,$$

and has a bounded invertible operator W^{-1} defined on $L^2(J,U)/\ker W$.

(A_3) **:** A nonlinear operator $g : J \times X \to Y$ satisfies

(i): For each $t \in J$, $g(t,z)$ is a continuous.

(ii): For each $z \in X$, $g(t,z)$ is a measurable.

(iii): There is a constant $\gamma (0 < \gamma < 1)$ and a function $h \in L^1(J)$ such that

$$|g(t,z)| \le h(t)|z|^\gamma$$

for all $(t,z) \in J \times X$.

Let $X = Y = L^2(0,\pi)$.

Define the operators $A, B : D(A)(\text{or} D(B)) \to X$ by

$$Az = -z_{xx}, \quad Bz = z - z_{xx}, \quad \text{respectively.}$$

where

$$\begin{aligned}
&D(A) = D(B)\\
&= \{z \in X : z, z_x \text{ are absolutely continuous } z_{xx} \in X, z(0) = z(\pi) = 0\}.
\end{aligned}$$

Define an operator $G : J \times X \to Y$ by

$$G(t,z)(x) = g(t, z_{xx}(x)).$$

The problem (4.1) can be formulated abstractly as:

$$\begin{aligned}
(Bz(t))' + Az(t) = Lu(t) + G(t, z(t)), \quad t \in J\\
z(0) = z_0.
\end{aligned}$$

We can written operators A and B, respectively, as

$$Az = \sum_{n=1}^{\infty} n^2(z, z_n)z_n, \quad z \in D(A),$$

and

$$Bz = \sum_{n=1}^{\infty}(1+n^2)(z, z_n)z_n, \quad z \in D(B),$$

where $z_n(x) = \sqrt{2/\pi}\sin nx,\ \ n = 1, 2, \cdots$ is the orthonormal set of eigenvectors of A. Moreover for $z \in X$, we get

$$B^{-1}z = \sum_{n=1}^{\infty}\frac{1}{1+n^2}(z, z_n)z_n,$$

$$-AB^{-1}z = \sum_{n=1}^{\infty} -\frac{n^2}{1+n^2}(z, z_n)z_n,$$

$$T(t)z = \sum_{n=1}^{\infty} e^{-n^2/(1+n^2)}(z, z_n)z_n.$$

It is easy to see that $-AB^{-1}$ generates a strongly continuous semigroup $T(t)$ on Y and $T(t)$ is compact such that $\|T(t)\| \leq e^{-t}$ for each $t > 0$. Also, the operator G satisfies (C_6) (see [6]). So all the conditions stated in the above theorem are satisfied. Consequently Eq.(4.1) is controllable on J.

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