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REDUCTION OF MODES FOR THE COMPUTATIONS OF NAVIER-STOKES EQUATIONS

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ABSTRACT. This article is a survey article for a reduced-order modeling approach for computation of time-dependent Navier-Stokes flows. A reduced-basis method is introduced. The choices for the reduced basis method are the Lagrange subspace, the Hermite subspace and the Taylor subspace. We then introduce the POD-based method and CVT-based method.

1. INTRODUCION

Optimal control problems that involve partial differential equations as state equations are formidable problems to solve in real time. One such situation arises in control of fluid dynamical systems in which the state equations are the Navier-Stokes equations. We will discuss some reduction-type method which may help to overcome this difficulty.

In order to illustrate the reduced-basis method, we consider the stationary Navier-Stokes equations

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{1.2}$$

with appropriate boundary conditions for $\nu \in \mathbb{R}$ and $\mathbf{u} \in \mathbf{X}$. The above problem is a parameterized one. The constant ν presents kinematic viscosity about which we choose to interpolate to obtain a reduced-finite-dimensional set of basis elements. In standard finite element approximations, one approximate $\mathbf{X}$ with a piecewise polynomial space. However, the choices for the reduced basis method are different.

The Lagrange subspace. In this case, the basis elements are solutions of the nonlinear problem under study at various parameter values ν_j. The reduced subspace is given by

$$\mathbf{X}_R = \text{span}\left\{\mathbf{u}^j | \mathbf{u}^j = \mathbf{u}(\nu_j),\ j = 1, \ldots, M\right\}$$

This kind of subspace was used to study structural problems in [1]. A possible advantage in this choice is that updating the basis elements can be done one basis vector at a time instead of generating the whole space.

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The Hermite subspace. In this case, the basis elements are solutions and their first derivative at various parameter values ν_j. The reduced subspace is given by

$$\mathbf{X}_R = \text{span}\left\{\mathbf{u}^j = \mathbf{u}(\nu_j),\ \text{and}\ \left.\frac{\partial \mathbf{u}}{\partial \nu}\right|_{\nu=\nu_j}, j = 1, \ldots, M\right\}$$

The Taylor subspace. In this case, one assume at some value of ν, say ν^*, the solution is known and it has M derivatives. The reduced subspace is given by

$$\mathbf{X}_R = \text{span}\left\{\mathbf{u}^j \mid \mathbf{u}^j = \left.\frac{\partial \mathbf{u}}{\partial \nu}\right|_{\nu=\nu^*}, j = 1, \ldots, M\right\}$$

where y^j is obtained from successive differentiation of (1.1)–(1.2). This choice has been extensively used in literature for structure analysis problems and for high Reynolds number steady state flow calculations [5, 6, 9].

2. The reduced basis method for Navier-Stokes equations

In this section, we introduce the reduced-basis method for Navier-Stokes equations. The Navier-Stokes equations are in primitive variabes

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \times (0, T] \tag{2.1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega \times (0, T] \tag{2.2}$$

$$\mathbf{u} = \mathbf{b} \quad \text{on } \partial\Omega \times (0, T] \tag{2.3}$$

$$\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \quad \text{in } \Omega, \tag{2.4}$$

where $\mathbf{u}(t, \mathbf{x})$ and $p(t, \mathbf{x})$ denote the velocity and pressure, respectively, $\mathbf{f}(t, \mathbf{x})$ is the body force per unit mass and $\mathbf{u}_0$ is the initial velocity. Furthermore, T is a positive constant, $\mathbf{b}$ is the boundary velocity, and Ω is a bounded region in $\mathbb{R}^2$ whose boundary is $\partial\Omega$.

2.1. Variational formulation.

We use a variational formulation and finite element method to approximate (2.1)–(2.4), but other methods can be also used with the reduced-basis method.

A variational formulation of the problem (2.1)–(2.4) is following: Find $\mathbf{u} \in L^2(0, T; \mathbf{V}_b)$ and $p \in L^2(0, T; L^2_0(\Omega))$ such that

$$\int_\Omega \mathbf{u}_t\cdot\mathbf{v}\,d\Omega + \nu\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\,d\Omega + \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\,d\Omega - \int_\Omega p\nabla\cdot\mathbf{v}\,d\Omega = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\Omega \quad \text{for all } \mathbf{v} \in \mathbf{H}^1_0(\Omega) \tag{2.5}$$

$$\int_\Omega q\nabla\cdot\mathbf{u} d\Omega = 0 \quad \text{for all } q \in L^2_0(\Omega) \tag{2.6}$$

$$\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \quad \text{in } \Omega, \tag{2.7}$$

where $\mathbf{V}_b = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \mathbf{u} = \mathbf{b} \text{ on } \partial\Omega, \mathbf{b} \in \mathbf{H}^{1/2}(\partial\Omega)\}$.

A typical finite element approximation of (2.5)–(2.6) is to seek solutions $\mathbf{u}^h(t,\cdot) \in \mathbf{V}_b^h \subset \mathbf{V}_b$ and $p^h \in S_0^h \subset L_0^2(\Omega)$,

$$\int_\Omega \mathbf{u}_t^h \cdot \mathbf{v}^h \, d\Omega + \nu \int_\Omega \nabla \mathbf{u}^h : \nabla \mathbf{v}^h \, d\Omega + \int_\Omega (\mathbf{u}^h \cdot \nabla)\mathbf{u}^h \cdot \mathbf{v}^h \, d\Omega$$

$$- \int_\Omega p^h \nabla \cdot \mathbf{v}^h \, d\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{v}^h \, d\Omega \quad \text{for all } \mathbf{v}^h \in \mathbf{V}_0^h \tag{2.8}$$

$$\int_\Omega q^h \nabla \cdot \mathbf{u}^h d\Omega = 0 \quad \text{for all } q^h \in S_0^h \tag{2.9}$$

where $\mathbf{V}_0^h \subset \mathbf{H}_0^1(\Omega)$ and $S_0^h \subset L_0^2(\Omega)$.

2.2. **The reduced-basis method and reduced-order model.** First, we introduce the steady state case [9]. Here we use some basis elements $\{\phi_i\}$ be given by $\{\phi_i = \mathbf{u}^h\}$. One can generate such as elements by solving

$$\nu \int_\Omega \nabla \mathbf{u}^h : \nabla \mathbf{v}^h \, d\Omega + \int_\Omega (\mathbf{u}^h \cdot \nabla)\mathbf{u}^h \cdot \mathbf{v}^h \, d\Omega - \int_\Omega p^h \nabla \cdot \mathbf{v}^h \, d\Omega$$

$$= \int_\Omega \mathbf{f} \cdot \mathbf{v}^h \, d\Omega \quad \text{for all } \mathbf{v}^h \in \mathbf{V}_0^h \tag{2.10}$$

$$\int_\Omega q^h \nabla \cdot \mathbf{u}^h d\Omega = 0 \quad \text{for all } q^h \in S_0^h \tag{2.11}$$

for different values of parameter ν. Thus given a set of values $\{\nu_i : i = 1, \ldots, M\}$, we solve (2.10)–(2.11) M times to determine the set $\{\hat{\mathbf{u}}_m : m = 1, \ldots, M\}$, where $\hat{\mathbf{u}}_i = \mathbf{u}^h(\nu_i)$. We then set

$$\mathbf{V}^M = \operatorname{span}\{\hat{\mathbf{u}}_i : i = 1, \ldots, M\} \subset \mathbf{V}^h.$$

Once we have a set of reduced basis functions we write the reduced-order model in the form : seek $u^M \in \mathbf{V}^M \subset V^h$ such that

$$\nu \int_\Omega \nabla \mathbf{u}^M : \nabla \mathbf{v}^M \, d\Omega + \int_\Omega (\mathbf{u}^M \cdot \nabla)\mathbf{u}^M \cdot \mathbf{v}^M \, d\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{v}^M \, d\Omega, \quad \forall \mathbf{v}^M \in \mathbf{V}_0^M, \tag{2.12}$$

where $\mathbf{V}_0^M = \mathbf{V}^M \cap \mathbf{V}_0^h$. Note that, by construction $\mathbf{u}^M$ automatically satisfies (2.11) and, due to the global support of the reduced-basis elements, the system (2.12) is equivalent to a dense lower order nonlinear system of equations as opposed to the system (2.10)–(2.11) which is a sparse nonlinear system due to the local support of the basis.

We now introduce the time-dependent case. Let the reduced-order basis elements $\{\phi_i\}$ be given by $\{\phi_i = \mathbf{u}^h(t_i, \cdot)\}$. We can generate such basis elements by three different ways. We generate basis elements by using an accurate numerical method with appropriate step size to integrate (2.8) in time, and then the first simple way is that we select the discrete time solution $\{\hat{\mathbf{u}}_i\}$ at a given set of values $\{t_i\}$, $i = 1, \ldots, M$, [4]. The other methods are the proper orthogonal decomposition [10, 11] and the centroidal Voronoi tessellation(CVT) method [2, 3].

We then obtain

$$\mathbf{V}^R = \operatorname{span}\{\hat{\mathbf{u}}_i : i = 1, \ldots, M\}.$$

The reduced-basis subspace $\mathbf{V}^R$ consists of the trial and test function where the test functions in $\mathbf{V}_0^R = \mathbf{V}^R \cap \mathbf{V}_0^h$ satisfy the homogeneous boundary condition. Once we have a set of reduced-basis functions we write the reduced-order model in the form : seek $\mathbf{u}^M(t,\cdot) \in \mathbf{V}^M = \text{span}\{\mathbf{u}_i \,:\, i = 1,\ldots,M\} \subset \mathbf{V}^h$ such that

$$\int_\Omega \mathbf{u}_t^M \cdot \mathbf{v}^M \, d\Omega + \nu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \, d\Omega + \int_\Omega (\mathbf{u}^M \cdot \nabla)\mathbf{u}^M \cdot \mathbf{v}^M \, d\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{v}^M \, d\Omega \quad \forall \mathbf{v}^M \in \mathbf{V}_0^M \tag{2.13}$$

$$(\mathbf{u}^M, \mathbf{v}^M)_{\partial\Omega} = (\mathbf{u}_b, \mathbf{v}^M)_{\partial\Omega} \quad \forall \mathbf{v}^M \in \mathbf{V}^M|_{\partial\Omega}, \tag{2.14}$$

and

$$(\mathbf{u}(0,\mathbf{x}), \mathbf{v}^M) = (\mathbf{u}_0(\mathbf{x}), \mathbf{v}^M) \quad \text{for all } \mathbf{v}^M \in \mathbf{V}_0^M, \tag{2.15}$$

where $\mathbf{V}_0^M = \mathbf{V}^M \cap \mathbf{V}_0^h$.

3. Computations of the reduced order model

One can generate basis elements $\{\phi_i\}_{i=1}^M \subset \mathbf{V}^M$ for the reduced-order model by Lagrange type method, POD or CVT method. Given the basis elements $\{\phi_i\}_{i=1}^M$, the reduced-order solution $\mathbf{u}^M$ is formed by setting

$$\mathbf{u}^M(t) = \sum_{i=1}^M \alpha_i(t)\phi_i, \tag{3.1}$$

The solution $\mathbf{u}^M$ is computed from

$$\int_\Omega \frac{\partial}{\partial t}\mathbf{u}^M \cdot \mathbf{v}^M \, d\Omega + \nu \int_\Omega \nabla \mathbf{u}^M : \nabla \mathbf{v}^M \, d\Omega + \int_\Omega (\mathbf{u}^M \cdot \nabla)\mathbf{u}^M \cdot \mathbf{v}^M \, d\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{v}^M \, d\Omega \quad \text{for all } \mathbf{v}^M \in \mathbf{V}_0^M \tag{3.2}$$

where $\mathbf{V}_0^M = \text{span}\{\boldsymbol{\phi}_i \,:\, i = 1,\ldots,M-1\}$ is the span of the test functions.

Let us rewrite (3.2) using the representation (3.1) with $\alpha_i(t) \in \mathbb{R}$. Using (3.1) and taking $\mathbf{v}^M = \boldsymbol{\phi}_j, j = 1,\ldots,M-1$, in (3.2), we get for $j = 1, dots, M-1$

$$\sum_{i=1}^{M-1} \frac{d}{dt}\alpha_i(t)(\boldsymbol{\phi}_i, \boldsymbol{\phi}_j) + \nu \sum_{i=1}^{M-1} \alpha_i(t)(\nabla\boldsymbol{\phi}_i, \nabla\boldsymbol{\phi}_j) + \left(\sum_{i=1}^{M-1} \alpha_i(t)\boldsymbol{\phi}_i \cdot \nabla \sum_{k=1}^{M-1} \alpha_k(t)\boldsymbol{\phi}_k, \boldsymbol{\phi}_j \right) = (\mathbf{f}, \boldsymbol{\phi}_j) \tag{3.3}$$

and

$$\sum_{i=1}^{M-1} \alpha_i(0)(\boldsymbol{\phi}_i, \boldsymbol{\phi}_j) = (\mathbf{u}_0, \boldsymbol{\phi}_j) \tag{3.4}$$

or equivalently, the nonlinear ordinary differential equations

$$\mathcal{M}\frac{d}{dt}\boldsymbol{\alpha}(t) + \mathcal{H}\boldsymbol{\alpha}(t) + (\boldsymbol{\alpha}(t))^T\mathcal{N}\boldsymbol{\alpha}(t) = \mathbf{F}(t) \tag{3.5}$$

where the mass matrix $\mathcal{M} = (\mathcal{M}_{ij})$, the stiffness matrix $\mathcal{H} = (\mathcal{H}_{ij})$, and the forcing term $\mathbf{F} = (F_i)$, $\boldsymbol{\alpha} = (\alpha_i)$:

$$\mathcal{M}_{ij} = \int_\Omega \boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_j \, d\Omega,$$

$$\mathcal{H}_{ij} = \nu \int_\Omega \nabla \boldsymbol{\phi}_i : \nabla \boldsymbol{\phi}_j \, d\Omega,$$

$$\mathbf{F}_i(t) = \int_\Omega \mathbf{F}(t) \cdot \boldsymbol{\phi}_i \, d\Omega.$$

We believe that this article is not sufficient to introduce the reduced-order type methods. For more details, one may refer the references listed in this article and references there in.

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