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OPTIMAL CONTROL FOR A PARABOLIC SYSTEM MODELLING CHEMOTAXIS

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ABSTRACT. We study the optimal control and robust control problems for a parabolic system modelling chemotaxis. That is, we obtain uniqueness of the optimal control under some conditions. Moreover, we consider the uniqueness of the saddle point.

1. INTRODUCTION

In this note we study the optimal control problem for the Keller-Segel equations:

$$\text{(P)} \qquad \text{Minimize} \quad J(u)$$

with the cost functional $J(u)$ of the form

$$J(u) = \int_0^T \|y(u) - y_d\|^2_{H^1(\Omega)} dt + \gamma \int_0^T \|u\|^2_{H^\varepsilon(\Omega)} dt, \quad u \in L^2(0,T; H^\varepsilon(\Omega)),$$

where $y = y(u)$ is governed by the Keller-Segel equations

$$\begin{aligned}
&\frac{\partial y}{\partial t} = a\Delta y - b\nabla\{y\nabla\rho\} \qquad \text{in } \Omega \times (0,T], \\
&\frac{\partial \rho}{\partial t} = d\Delta\rho + fy - g\rho + u \qquad \text{in } \Omega \times (0,T], \\
&\frac{\partial y}{\partial n} = \frac{\partial \rho}{\partial n} = 0 \qquad \text{on } \partial\Omega \times (0,T], \\
&y(x,0) = y_0(x), \quad \rho(x,0) = \rho_0(x) \qquad \text{in } \Omega.
\end{aligned} \tag{1.1}$$

Here, Ω is a bounded region in $\mathbf{R^2}$ of $\mathcal{C}^3$ class. a, b, d, f, g are given positive numbers and γ is a given nonnegative number. $u \geq 0$ is a control function in some bounded subsets. ε is some fixed exponent such that $0 < \varepsilon < 1/2$. $n = n(x)$ is the outer normal vector at a boundary point $x \in \partial\Omega$ and $\frac{\partial}{\partial n}$ denotes the differentiation along the vector n. $y_0(x)$ and $\rho_0(x)$ are nonnegative initial functions in $L^2(\Omega)$ and in $H^{1+\varepsilon}(\Omega)$, respectively. y, ρ are unknown functions of the Cauchy problem (1.1).

The Keller-Segel equations (1.1) was introduced by Keller and Segel [4] to describe the aggregation process of the cellular slime molds by chemical attraction. Unknown functions $y = y(x,t)$ and $\rho = \rho(x,t)$ denote the concentration of amoebae in Ω at time t and the concentration of chemical substance in Ω at time t, respectively. The chemotactic term $-b\nabla \cdot \{y\nabla\rho\}$ indicates that the cells are sensitive to

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chemicals and are attracted by them, and the production term fy indicates that the chemical substance is itself emitted by cells.

Optimal control problem associated to nonlinear equations have already studied by many authors ([1], [2], [3], [5]). Recently, Ryu and Yagi [5] studied the distributed optimal control problem for Keller-Segel equations of non-monotone type. Under the mild assumptions, this paper obtains the uniqueness of the optimal control. We also consider the robust control problem as a differential game finding the best control which takes into account the worst disturbance.

2. Mathematical setting

Let us briefly recall the way how to formulate (1.1) as a semilinear abstract differential equation in a Hilbert space. Let $A_1 = -a\Delta + a$ and $A_2 = -d\Delta + g$ be the Laplace operators equipped with the Neumann boundary conditions. The part of A_i in $L^2(\Omega)$ is a positive definite self-adjoint operator in $L^2(\Omega)$ with the domain $\mathcal{D}(A_i) = H^2_n(\Omega) = \{y \in H^2(\Omega); \frac{\partial y}{\partial n} = 0 \text{ on } \partial\Omega\}$. $\mathcal{D}(A_i^\theta) = H^{2\theta}(\Omega)$ for $0 \le \theta < \frac{3}{4}$, and $\mathcal{D}(A_i^\theta) = H^{2\theta}_n(\Omega)$ for $\frac{3}{4} < \theta \le \frac{3}{2}$ (see Triebel [8]).

We introduce two product Hilbert spaces $\mathcal{V} \subset \mathcal{H}$ as $\mathcal{V} = H^1(\Omega)\times\mathcal{D}(A_2^{1+\varepsilon/2})$ and $\mathcal{H} = L^2(\Omega) \times \mathcal{D}(A_2^{(1+\varepsilon)/2})$, respectively, where ε is some fixed exponent $\varepsilon \in (0, \frac{1}{2})$. By the identification of $\mathcal{H}$ and its dual $\mathcal{H}'$, we have: $\mathcal{V} \subset \mathcal{H} = \mathcal{H}' \subset \mathcal{V}'$. It is then seen that $\mathcal{V}' = (H^1(\Omega))' \times \mathcal{D}(A_2^{\varepsilon/2})$. The norms of $\mathcal{V}$, $\mathcal{H}$, and $\mathcal{V}'$ are denoted by $\|\cdot\|$, $|\cdot|$, and $\|\cdot\|_*$, respectively. The duality product between $\mathcal{V}$ and $\mathcal{V}'$ is denoted by $\langle\cdot,\cdot\rangle$.

We set a symmetric sesquilinear form on $\mathcal{V}\times\mathcal{V}$:

$$a(Y, \widetilde{Y}) = \big(A_1^{1/2} y, A_1^{1/2}\tilde{y}\big)_{L^2} + \big(A_2^{1+\varepsilon/2}\rho, A_2^{1+\varepsilon/2}\tilde{\rho}\big)_{L^2}, \qquad Y = \begin{pmatrix} y \\ \rho \end{pmatrix},\ \widetilde{Y} = \begin{pmatrix} \tilde{y} \\ \tilde{\rho} \end{pmatrix} \in \mathcal{V}.$$

Obviously, the form satisfies

$$|a(Y, \widetilde{Y})| \le M\|Y\|\|\widetilde{Y}\|, \quad Y, \widetilde{Y} \in \mathcal{V}, \tag{2.1}$$

$$a(Y, Y) \ge \delta\|Y\|^2, \quad Y \in \mathcal{V} \tag{2.2}$$

with some δ and $M > 0$. This form then defines a linear isomorphism $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ from $\mathcal{V}$ to $\mathcal{V}'$, and the part of A in $\mathcal{H}$ is a positive definite self-adjoint operator in $\mathcal{H}$ with the domain $\mathcal{D}(A) = \mathcal{D}(A_1) \times \mathcal{D}(A_2^{(3+\varepsilon)/2})$.

(1.1) is, then, formulated as an abstract equation

$$\frac{dY}{dt} + AY = F(Y) + U(t), \quad 0 < t \le T, \qquad Y(0) = Y_0 \tag{2.3}$$

in the space $\mathcal{V}'$. Here, $F(\cdot) : \mathcal{V} \to \mathcal{V}'$ is the mapping

$$F(Y) = \begin{pmatrix} -b\nabla\{y\nabla\rho\} + ay \\ fy \end{pmatrix}, \qquad Y = \begin{pmatrix} y \\ \rho \end{pmatrix} \in \mathcal{V}.$$

$U(t) = \binom{0}{u(t)}$ and $Y_0 = \binom{y_0}{\rho_0}$.

As verified in [5, Sec. 2], $F(\cdot)$ satisfies the following conditions:

(f.i) For each $\eta > 0$, there exists an increasing continuous function $\phi_\eta : [0,\infty) \to [0,\infty)$ such that

$$\|F(Y)\|_* \le \eta\|Y\| + \phi_\eta(|Y|), \quad Y \in \mathcal{V};$$

(f.ii) For each $\eta > 0$, there exists an increasing continuous function $\psi_\eta : [0,\infty) \to [0,\infty)$ such that

$$\begin{aligned}&\|F(\widetilde{Y}) - F(Y)\|_* \\ &\qquad \le \eta\|\widetilde{Y} - Y\| + (\|\widetilde{Y}\| + \|Y\| + 1)\psi_\eta(|\widetilde{Y}| + |Y|)|\widetilde{Y} - Y|, \quad \widetilde{Y}, Y \in \mathcal{V}.\end{aligned}$$

Furthermore, $F(Y)$ is the first-order Fréchet differentiable with the derivative

$$F'(Y)Z = \begin{pmatrix} -b\nabla\{y\nabla w\} - b\nabla\{z\nabla\rho\} + az \\ fz \end{pmatrix},$$

$F'(\cdot)$ satisfies the following estimates:

(f.iii) For each $\eta > 0$, there exists an increasing continuous functions $\mu_\eta, \nu : [0,\infty) \to [0,\infty)$ such that

$$|\langle F'(Y)Z, P\rangle| \le \begin{cases} \eta\|Z\|\|P\| + (\|Y\| + 1)\mu_\eta(|Y|)|Z|\|P\|, & Y, Z, P \in \mathcal{V}, \\ \eta\|Z\|\|P\| + (\|Y\| + 1)\mu_\eta(|Y|)\|Z\||P|, & Y, Z, P \in \mathcal{V}, \\ \nu(|Y|)\|Z\|\|P\|, \quad Y, Z, P \in \mathcal{V}. \end{cases}$$

(f.iv) $F'(\cdot)$ is continuous from $\mathcal{H}$ into $\mathcal{L}(\mathcal{V}, \mathcal{V}')$.

We then obtain the following result.

Theorem 2.1 ([5, Theorem 2.1])**.** *Let* (2.1), (2.2), (f.i), *and* (f.ii) *be satisfied. Then, for any $U \in L^2(0,T;\mathcal{V}')$ and $Y_0 \in \mathcal{H}$, there exists a unique weak solution*

$$Y \in H^1(0, T(Y_0,U); \mathcal{V}') \cap \mathcal{C}([0, T(Y_0,U)]; \mathcal{H}) \cap L^2(0, T(Y_0,U); \mathcal{V})$$

to (2.3), *the number $T(Y_0,U) > 0$ is determined by the norms $\|U\|_{L^2(0,T;\mathcal{V}')}$ and $|Y_0|$.*

3. Distributed control problem

Let $\mathcal{U} = L^2(0,T;\mathcal{V}')$ and $\mathcal{U}_{ad}$ be closed, bounded and convex subset of $\mathcal{U}$. The problem (P) is obviously formulated as follows:

$$(\overline{P}) \qquad \text{minimize} \;\; J(U),$$

where the cost functional $J(U)$ is of the form

$$J(U) = \int_0^S \|DY(U) - Y_d\|^2 dt + \gamma \int_0^S \|U\|_*^2 dt, \quad U \in \mathcal{U}_{ad}.$$

Here, $Y(U)$, $U \in \mathcal{U}_{ad}$, is the weak solution of (2.3) and is assumed to exist on a fixed interval $[0,S]$. $D\binom{y}{\rho} = \binom{y}{0}$ is a bounded operator from $\mathcal{V}$ into $\mathcal{V}$ and $Y_d = \binom{y_d}{0}$ is a fixed element of $L^2(0,S;\mathcal{V})$ with $y_d \in L^2(0,T;H^1(\Omega))$. γ is a nonnegative constant.

Theorem 3.1 ([5, Theorem 4.1])**.** *There exists an optimal control $\overline{U} \in \mathcal{U}_{ad}$ for $(\overline{P})$ such that $J(\overline{U}) = \min_{U \in \mathcal{U}_{ad}} J(U)$.*

To derive the uniqueness of optimal control for $(\overline{P})$, the second order Fréchet derivative of the mapping $F(\cdot) : \mathcal{V} \to \mathcal{V}'$ is necessary. It is indeed observed by a direct calculation that

$$F''(Y)(Z,Z) = \begin{pmatrix} -2b\nabla\{z\nabla w\} \\ 0 \end{pmatrix}, \quad Y = \begin{pmatrix} y \\ \rho \end{pmatrix},\ Z = \begin{pmatrix} z \\ w \end{pmatrix} \in \mathcal{V}.$$

and the following estimate:

(f.v) There exists $N > 0$ such that

$$\|F''(Y)(Z,Z)\|_* \leq N|Z|\|Z\|, \qquad Y, Z \in \mathcal{V}.$$

Proposition 3.2. *The mapping $Y : \mathcal{U}_{ad} \to H^1(0,S;\mathcal{V}') \cap \mathcal{C}([0,S];\mathcal{H}) \cap L^2(0,S;\mathcal{V})$ is Gâteaux differentiable with respect to U. For $V \in \mathcal{U}_{ad}$, $Y'(U)V = Z$ is the unique solution in $H^1(0,S;\mathcal{V}') \cap \mathcal{C}([0,S];\mathcal{H}) \cap L^2(0,S;\mathcal{V})$ of the problem*

$$\frac{dZ}{dt} + AZ - F'(Y)Z = V(t), \quad 0 < t \leq S,$$
$$Z(0) = 0.$$

Moreover, there exists $\bar{\gamma}$ such that, for $\gamma > \bar{\gamma}$, the mapping $U \to J(U)$ is strictly convex.

Proof. The strict convexity of J as well as the existence of the Gâteaux derivatives are obtained in [6].

The main result is given by:

Theorem 3.3. *For all $\gamma > \bar{\gamma}$, there exists a unique optimal control $\overline{U} \in \mathcal{U}_{ad}$ for $(\overline{P})$.*

Remark 3.4. *We can also obtain the strict convexity of the cost functional and the uniqueness of the optimal control if one can assume that S is sufficiently small.*

4. Robust control problem

The cost functional contains an additional term due to disturbance and it is given by:

$$J(U,\Lambda) = \int_0^S \|DY(U,\Lambda) - Y_d\|^2 dt + \int_0^S [\gamma\|U\|_*^2 - l\|\Lambda\|_*^2] dt.$$

Here, $Y(U,\Lambda)$ is the weak solution of

$$\frac{dY}{dt} + AY = F(Y) + U(t) + \Lambda(t), \quad 0 < t \leq T,$$
$$Y(0) = Y_0.$$

$U(t) = \binom{0}{u(t)}$ and $\Lambda(t) = \binom{0}{\lambda(t)}$. γ and l are nonnegative constants. As in the case of the optimal control problem, we assume that $\mathcal{U}_{ad}$ and $\mathcal{G}_{ad}$ are closed, bounded, and convex subsets of $L^2(0,S;\mathcal{V}')$.

The main result is as follows (For the detailed proof, we refer to [7]).

Theorem 4.1. *There exist $\bar{\gamma}$ and $\bar{l}$ such that for $\gamma > \bar{\gamma}$ and $l > \bar{l}$, $U \to J(U,\Lambda)$ is strictly convex lower semicontinuous and $\Lambda \to J(U,\Lambda)$ is strictly concave upper* *semicontinuous. Moreover, there exists a unique saddle point* $(\overline{U}, \overline{\Lambda}) \in \mathcal{U}_{ad} \times \mathcal{G}_{ad}$ *such that*

$$J(\overline{U}, \Lambda) \leq J(\overline{U}, \overline{\Lambda}) \leq J(U, \overline{\Lambda}) \quad \forall (U, \Lambda) \in \mathcal{U}_{ad} \times \mathcal{G}_{ad}.$$

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