Trends in Mathematics
Information Center for Mathematical Sciences
Volume 6, Number 1, June, 2003, Pages 41–43

SOLVING n-TH ORDER ORDINARY DIFFERENTIAL EQUATIONS

HYUN-MIN KIM

Abstract. For solving the second order ordinary differential equation it is necessary to solve the quadratic eigenvalue problem. We show how solving the quadratic matrix equation offers a potential saving of work and storage in numerical sense.

1. The n-th order ordinary differential equation

The n-th order ordinary differential equation can be defined by

$$A_n \frac{d^n}{dt^n} x(t) + A_{n-1} \frac{d^{n-1}}{dt^{n-1}} x(t) + \cdots + A_1 \frac{d}{dt} x(t) + A_0 x(t) = 0, \tag{1.1}$$

where $A_n,\ A_{n-1},\ \ldots,\ A_0$ are $n \times n$ complex matrices. For solving the equation (1.1) we need to solve the polynomial eigenvalue problem

$$P(\lambda)v = (\lambda^n A_n + \lambda^{n-1} A_{n-1} + \cdots + \lambda A_1 + A_0)v = 0.$$

In this paper we consider specially how the second order ordinary differential equation can be computed.

2. The second order ordinary differential equation

Figure 2.1 illustrates a connected damped mass-spring system. The i-th mass of weight m_i is connected to to the $(i+1)$-th mass by a spring with constant k_i and damper with constant d_i, and is also connected to the ground by a spring with constant κ_i and damper constant τ_i [9]. The vibration of this system gives a second order differential equation

$$A_2 \frac{d^2}{dt^2} x(t) + A_1 \frac{d}{dt} x(t) + A_0 x(t) = 0, \tag{2.1}$$

where the mass matrix $A_2 = \mathrm{diag}(m_1, \ldots, m_n)$ is diagonal and the damping matrix A_1 and stiffness matrix A_0 are symmetric tridiagonal. The general solution of the equation (2.1) can be expressed by

$$x(t) = cWe^{\Lambda t}, \qquad W = [w_1, \ldots, w_n], \qquad \Lambda = \mathrm{diag}(\mu_i),$$

where the pairs $(\mu_i, w_i)_{i=1}^n$ are chosen from pair $(\lambda_i, v_i)_{i=1}^{2n}$ which satisfies the quadratic eigenvalue problem

$$Q(\lambda)v = (\lambda^2 A_2 + \lambda A_1 + A_0)v = 0, \qquad \lambda \in \mathbb{C}, \qquad v \in \mathbb{C}^n. \tag{2.2}$$

2000 *Mathematics Subject Classification.* 55F30, 65H10.

Key words and phrases. polynomial eigenvalue problem, quadratic matrix equation, quadratic eigenvalue problem.

PSfrag replacements
x_1
x_i
x_n
k_1
k_{i-1}
k_i
k_{n-1}
m_1
m_i
m_n
d_1
d_{i-1}
d_i
d_{n-1}
κ_1
κ_i
κ_n
τ_1
τ_i
τ_n

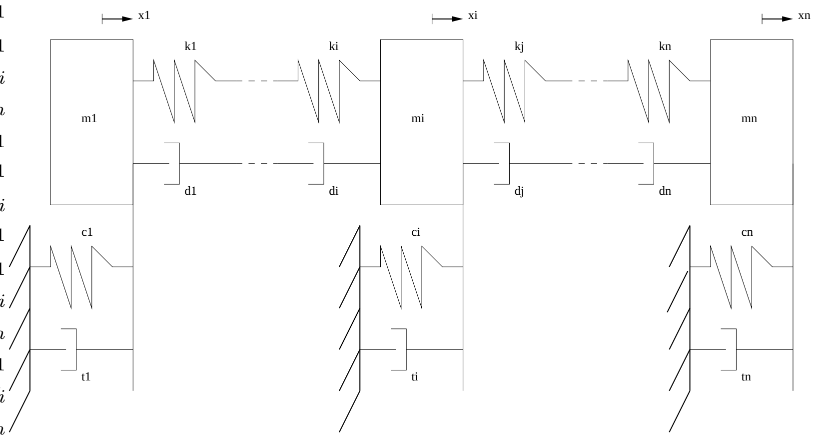

FIGURE 2.1. An n degree of freedom damped mass-spring system.

A standard approach for solving the quadratic eigenvalue problem is to convert (2.2) to a generalized eigenvalue problem of twice the dimension, $2n$.

There are several possible ways and three special reductions have been examined by Tisseur [8]. Setting $\hat{x} = \lambda x$, we get

$$\begin{cases} \lambda x - \hat{x} &= 0, \\ \lambda A_2 \hat{x} + A_0 x + A_1 \hat{x} &= 0. \end{cases}$$

and then the equation can be expressed by

$$\lambda \begin{bmatrix} I_n & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} 0 & -I_n \\ A_0 & A_1 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} = 0.$$

Thus, we obtain a generalized eigenvalue problem:

$$\begin{bmatrix} 0 & I_n \\ -A_0 & -A_1 \end{bmatrix} y = \lambda \begin{bmatrix} I_n & 0 \\ 0 & A_2 \end{bmatrix} y$$

with

$$y = \begin{bmatrix} x \\ \lambda x \end{bmatrix}.$$

Now, suppose that a solvent S of the quadratic matrix equation,

$$Q(X) = A_2 X^2 + A_1 X + A_0 = 0, \qquad A_2, A_1, A_0, X \in \mathbb{C}^{n \times n}, \tag{2.3}$$

can be found. The following result gives an important role of a solvent S in the quadratic eigenvalue problem.

Theorem 2.1. [4, Cor. 3.6], [7, Thm. 3.3] *When $Q(\lambda)$ in (2.2) is divided on the right by $X - \lambda I$ the remainder is $A_2X^2 + A_1X + A_0$, and when $Q(\lambda)$ is divided on the left by $X - \lambda I$ the remainder is $X^2A_2 + XA_1 + A_0$.*

From the Theorem 2.1, the quadratic eigenvalue problem $Q(\lambda)$ can be factorized in a simple form at a solvent S of $Q(X)$:

$$Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0 = -(A_1 + A_2 S + \lambda A_2)(S - \lambda I).$$

Hence the problem is reduced to solving $n \times n$ eigenproblems: that of S and the generalized eigenvalue problem $(A_1 + A_2 S)x = -\lambda A_2 x$. (This approach can be used in the solution of differential eigenproblems [1].) This means if S can be found by working only with $n \times n$ matrices then this approach offers a potential saving of work and storage.

Recently several authors [2], [3], [5], [6] introduced some numerical methods for solving the quadratic matrix equation $Q(X)$ in (2.3).

References

[1] T. J. Bridges and P. J. Morris. Differential eigenvalue problems in which the parameter appears nonlinearly. *J. Comp. Phys.*, 55:437–460, 1984.

[2] George J. Davis. Numerical solution of a quadratic matrix equation. *SIAM J. Sci. Stat. Comput.*, 2(2):164–175, 1981.

[3] George J. Davis. Algorithm 598: An algorithm to compute solvents of the matrix equation $AX^2 + BX + C = 0$. *ACM Trans. Math. Software*, 9(2):246–254, 1983.

[4] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix Polynomials*. Academic Press, New York, 1982. xiv+409 pp. ISBN 0-12-287160-X.

[5] Nicholas J. Higham and Hyun-Min Kim. Numerical analysis of a quadratic matrix equation. *IMA J. Numer. Anal.*, 20(4):499–519, 2000.

[6] Nicholas J. Higham and Hyun-Min Kim. Solving a quadratic matrix equation by Newton's method with exact line searches. *SIAM J. Matrix Anal. Appl.*, 23(2):303–316, 2001.

[7] Peter Lancaster. *Lambda-Matrices and Vibrating Systems*. Pergamon Press, Oxford, 1966. xiii+196 pp.

[8] Françoise Tisseur. Backward error and condition of polynomial eigenvalue problems. *Linear Algebra and Appl.*, 309:339–361, 2000.

[9] Françoise Tisseur and Nicholas J. Higham. Structured pseudospectra for polynomial eigenvalue problems, with applications. *SIAM J. Matrix Anal. Appl.*, 23(1):187–208, 2001.

Department of Mathematics, Pusan National University, 30 Changjeon-dong, Keumjeong-ku, Pusan 609-735, South Korea

E-mail address: hyunmin@pusan.ac.kr