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NIELSEN NUMBERS ON INFRA-NILMANIFOLDS

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ABSTRACT. Let M be an infra-nilmanifold and $f : M \longrightarrow M$ be a self-map. Suppose M_K is a regular covering of M which is a compact nilmanifold with $\pi_1(M_K) = K$. Assume that $f_*(K) \subset K$. Then f has a lifting $\bar{f} : M_K \longrightarrow M_K$. We prove the following averaging formula for Nielsen numbers

$$N(f) = \frac{1}{[\pi_1(M) : K]} \sum_{\bar{\alpha} \in \pi_1(M)/K} N(\bar{\alpha}\bar{f}).$$

We also prove that $N(f) \geq N_K(f)$, and equality occurs if and only if every essential mod K fixed point class coincides with an ordinary fixed point class of f.

Notational conventions:

X a compact connected space,
$f : X \longrightarrow X$ a self-map,
π the group of covering transformations on the universal cover $\tilde{X}$ of X,
K a normal subgroup of π of finite index so that $f_*(K) \subset K$,
$\bar{X} = \tilde{X}/K$,
$p : \tilde{X} \longrightarrow X$, $p' : \tilde{X} \longrightarrow \bar{X}$, $\bar{p} : \bar{X} \longrightarrow X$; $p = \bar{p} \circ p'$,
$\bar{f} : \bar{X} \longrightarrow \bar{X}$ a fixed lifting of f on $\bar{X}$,
$\tilde{f} : \tilde{X} \longrightarrow \tilde{X}$ a fixed lifting of $\bar{f}$ on $\tilde{X}$,
$\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\}$ the fixed point set of the map $f : X \longrightarrow X$,
$\mathrm{fix}(\varphi) = \{\alpha \in \Pi \mid \varphi(\sigma) = \sigma\}$ the subgroup of Π fixed by $\varphi : \Pi \longrightarrow \Pi$.

Note that any lifting of f on $\tilde{X}$ is of the form $\alpha\tilde{f}$ where $\alpha \in \pi$, and any lifting of f on $\bar{X}$ is of the form $\bar{\alpha}\bar{f}$ where $\bar{\alpha} \in \pi/K$.

1. Decomposition of the Fixed point set

For the fixed liftings $\bar{f}$ and $\tilde{f}$, we have homomorphisms

$$\bar{\varphi} : \pi/K \longrightarrow \pi/K \quad \text{defined by } \bar{f}\bar{\alpha} = \bar{\varphi}(\bar{\alpha})\bar{f},$$
$$\varphi : \pi \longrightarrow \pi \quad \text{defined by } \tilde{f}\alpha = \varphi(\alpha)\tilde{f},$$

so that $\varphi' = \varphi|_K : K \longrightarrow K$ and the following diagram is commutative:

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This is a brief review of [3].

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \xrightarrow{i} & \pi & \xrightarrow{q} & \pi/K & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \bar{\varphi}} & & \\ 1 & \longrightarrow & K & \xrightarrow{i} & \pi & \xrightarrow{q} & \pi/K & \longrightarrow & 1. \end{array}$$

The homomorphism $\varphi : \pi \longrightarrow \pi$ defines the *Reidemeister action* of π on π as follows:

$$\pi \times \pi \longrightarrow \pi, \quad (\gamma, \alpha) \mapsto \gamma\alpha\varphi(\gamma)^{-1}.$$

Similarly, the homomorphisms $\varphi' : K \longrightarrow K$ and $\bar{\varphi} : \pi/K \longrightarrow \pi/K$ define the Reidemeister actions of K on K and π/K on π/K, respectively. Denote the sets of Reidemeister classes of K, π, π/K determined by φ', φ, $\bar{\varphi}$ by $\mathcal{R}[\varphi']$, $\mathcal{R}[\varphi]$, $\mathcal{R}[\bar{\varphi}]$, respectively.

The fixed point classes of f are the subsets $p\,\mathrm{Fix}(\alpha\tilde{f})$, $\alpha \in \pi$. Each fixed point class $p\,\mathrm{Fix}(\alpha\tilde{f})$ is determined by the Reidemeister class $[\alpha] \in \mathcal{R}[\varphi]$. The fixed point set $\mathrm{Fix}(f)$ splits into a disjoint union of fixed point classes.

Now we have an exact sequence of sets

$$\mathcal{R}[\varphi'] \xrightarrow{\hat{i}} \mathcal{R}[\varphi] \xrightarrow{\hat{q}} \mathcal{R}[\bar{\varphi}] \longrightarrow 1,$$

i.e., $\hat{q}$ is surjective and $\hat{q}^{-1}([\bar{1}]) = \mathrm{im}(\hat{i})$. For each $\bar{\alpha} \in \pi/K$ and $\alpha \in q^{-1}(\bar{\alpha})$, $\alpha\tilde{f}$ is a lifting of $\bar{\alpha}\bar{f}$ and f, and $\bar{\alpha}\bar{f}$ is a lifting of f. They induce homomorphisms $\tau_\alpha\varphi$, $\tau_{\bar{\alpha}}\bar{\varphi}$ and $\tau_\alpha\varphi'$, where τ_α denotes the conjugation by α, i.e.,

$$\tau_\alpha(\beta) = \alpha\beta\alpha^{-1}.$$

Then the following diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \xrightarrow{i_\alpha} & \pi & \xrightarrow{q_\alpha} & \pi/K & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \tau_\alpha\varphi'} & & \downarrow{\scriptstyle \tau_\alpha\varphi} & & \downarrow{\scriptstyle \tau_{\bar{\alpha}}\bar{\varphi}} & & \\ 1 & \longrightarrow & K & \xrightarrow{i_\alpha} & \pi & \xrightarrow{q_\alpha} & \pi/K & \longrightarrow & 1 \end{array}$$

is commutative, and the following sequence of groups

$$1 \longrightarrow \mathrm{fix}(\tau_\alpha\varphi') \xrightarrow{i_\alpha} \mathrm{fix}(\tau_\alpha\varphi) \xrightarrow{q_\alpha} \mathrm{fix}(\tau_{\bar{\alpha}}\bar{\varphi})$$

is exact, and the following sequence of sets

$$\mathcal{R}[\tau_\alpha\varphi'] \xrightarrow{\hat{i}_\alpha} \mathcal{R}[\tau_\alpha\varphi] \xrightarrow{\hat{q}_\alpha} \mathcal{R}[\tau_{\bar{\alpha}}\bar{\varphi}] \longrightarrow 1$$

is exact.

Lemma 1.1. *For each $k \in K$ and $\alpha \in \pi$, we have*

(1) $|\hat{q}^{-1}([\bar{\alpha}])| = |\hat{q}_\alpha^{-1}([\bar{1}])|$,
(2) $|\mathcal{R}[\varphi]| = \sum_{[\bar{\alpha}] \in \mathcal{R}[\bar{\varphi}]} |\hat{q}^{-1}([\bar{\alpha}])| = \sum_{[\bar{\alpha}] \in \mathcal{R}[\bar{\varphi}]} |\hat{q}_\alpha^{-1}([\bar{1}])| = \sum_{[\bar{\alpha}] \in \mathcal{R}[\bar{\varphi}]} |\mathrm{im}(\hat{i}_\alpha)|$,
(3) $|\mathcal{R}[\tau_\alpha\varphi']| = \sum_{[k] \in \mathrm{im}(\hat{i}_\alpha)} |\hat{i}_\alpha^{-1}([k])|$,
(4) $|\hat{i}_\alpha^{-1}([k])| = [\mathrm{fix}(\tau_{\bar{\alpha}}\bar{\varphi}) : q_{k\alpha}\,\mathrm{fix}(\tau_{k\alpha}\varphi)]$,
(5) $|\mathcal{R}[\tau_\alpha\varphi']| = \sum_{[k] \in \mathrm{im}(\hat{i}_\alpha)} [\mathrm{fix}(\tau_{\bar{\alpha}}\bar{\varphi}) : q_{k\alpha}\,\mathrm{fix}(\tau_{k\alpha}\varphi)]$.

The fixed point classes of f are labelled by the Reidemeister classes $\mathcal{R}[\varphi]$ of φ. We may relabel them in terms of the Reidemeister classes $\mathcal{R}[\bar\varphi]$ and $\mathcal{R}[\tau_\alpha\varphi']$. This relabelling is useful in comparing the Nielsen number $N(f)$ of f with the Nielsen numbers $N(\bar\alpha\bar f)$ of $\bar\alpha\bar f$.

Lemma 1.2 (Decompositions of the Fixed Point Sets)**.** *Let $f : X \longrightarrow X$ be a self-map on a compact connected space X. Then*

(1) $\bar p\operatorname{Fix}(\bar\alpha\bar f) = \coprod_{[k]\in\operatorname{im}(\hat i_\alpha)} p\operatorname{Fix}(k\alpha\tilde f)$,

(2) $\operatorname{Fix}(f) = \coprod_{[\bar\alpha]\in\mathcal{R}[\bar\varphi]}\ \coprod_{[k]\in\operatorname{im}(\hat i_\alpha)} p\operatorname{Fix}(k\alpha\tilde f)$.

Remark 1.3. The group π/K acts on the set $[\bar\alpha]\in\mathcal{R}[\bar\varphi]$ by the rule $\bar\alpha\mapsto\bar\beta\bar\alpha\bar\varphi(\bar\beta)^{-1}$. This action is transitive. The isotropy subgroup is

$$\{\bar\beta \mid \bar\beta\bar\alpha\bar\varphi(\bar\beta)^{-1}=\bar\alpha\} = \operatorname{fix}(\tau_{\bar\alpha}\bar\varphi).$$

Hence $[\pi : K] = |[\bar\alpha]|\cdot|\operatorname{fix}(\tau_{\bar\alpha}\bar\varphi)|$.

Remark 1.4. For each $x\in\bar p\operatorname{Fix}(\bar\alpha\bar f)$, we fix $\bar x\in\operatorname{Fix}(\bar\alpha\bar f)$ with $\bar p(\bar x)=x$. Then $\operatorname{Fix}(\bar\alpha\bar f)\cap\bar p^{-1}(x) = \{\bar\beta\bar x \mid \bar\beta\in\operatorname{fix}(\tau_{\bar\alpha}\bar\varphi)\}$. That is, there are $|\operatorname{fix}(\tau_{\bar\alpha}\bar\varphi)|$ points in $\operatorname{Fix}(\bar\alpha\bar f)\cap\bar p^{-1}(x)$. Moreover,

$$\begin{aligned} L(\bar\alpha\bar f) &= \sum_{\bar x\in\operatorname{Fix}(\bar\alpha\bar f)}\operatorname{ind}(\bar\alpha\bar f,\bar x) = |\operatorname{fix}(\tau_{\bar\alpha}\bar\varphi)|\sum_{x\in\bar p\operatorname{Fix}(\bar\alpha\bar f)}\operatorname{ind}(f,x)\\ &= |\operatorname{fix}(\tau_{\bar\alpha}\bar\varphi)|\cdot\operatorname{ind}(f,\bar p\operatorname{Fix}(\bar\alpha\bar f)).\end{aligned}$$

Lemma 1.5. *Let $k\in K$ and $\alpha\in\pi$. If $x\in p\operatorname{Fix}(k\alpha\tilde f)$, then there are $|q_{k\alpha}\operatorname{fix}(\tau_{k\alpha}\varphi)|$ points in $p'\operatorname{Fix}(k\alpha\tilde f)\cap\bar p^{-1}(x)$. Hence,*

$$|p'\operatorname{Fix}(k\alpha\tilde f)| = |q_{k\alpha}\operatorname{fix}(\tau_{k\alpha}\varphi)|\cdot|p\operatorname{Fix}(k\alpha\tilde f)|.$$

Remark 1.6. The projections p' and p are local homeomorphisms, and the index is a local invariant. Thus by Lemma 1.5, $p'\operatorname{Fix}(k\alpha\tilde f)$ is an essential fixed point class of $\bar\alpha\bar f$ if and only if $p\operatorname{Fix}(k\alpha\tilde f)$ is an essential fixed point class of f.

In all, we have the following natural one-to-one correspondences:

$$\begin{aligned} &\coprod_{\bar\alpha\in\pi/K}\ \coprod_{[k]\in\mathcal{R}[\tau_\alpha\varphi']} p'\operatorname{Fix}(k\alpha\tilde f)\\ \longleftrightarrow\quad &\coprod_{\bar\alpha\in\pi/K}\ \coprod_{[k]\in\mathcal{R}[\tau_\alpha\varphi']} |q_{k\alpha}\operatorname{fix}(\tau_{k\alpha}\varphi)|\cdot p\operatorname{Fix}(k\alpha\tilde f)\\ \longleftrightarrow\quad &[\pi : K]\coprod_{\bar\alpha\in\pi/K}\ \coprod_{[k]\in\mathcal{R}[\tau_\alpha\varphi']}\frac{|q_{k\alpha}\operatorname{fix}(\tau_{k\alpha}\varphi)|}{|[\bar\alpha]|\cdot|\operatorname{fix}(\tau_{\bar\alpha}\bar\varphi)|}p\operatorname{Fix}(k\alpha\tilde f)\ \text{by Remark 1.3}\\ \longleftrightarrow\quad &[\pi : K]\coprod_{\bar\alpha\in\pi/K}\frac{1}{|[\bar\alpha]|}\ \coprod_{[k]\in\mathcal{R}[\tau_\alpha\varphi']}\frac{1}{|\operatorname{fix}(\tau_{\bar\alpha}\bar\varphi) : q_{k\alpha}\operatorname{fix}(\tau_{k\alpha}\varphi)|}p\operatorname{Fix}(k\alpha\tilde f)\\ \longleftrightarrow\quad &[\pi : K]\coprod_{[\bar\alpha]\in\mathcal{R}[\bar\varphi]}\ \coprod_{[k]\in\operatorname{im}(\hat i_\alpha)} p\operatorname{Fix}(k\alpha\tilde f)\ \text{by Lemma 1.1.}\end{aligned}$$

2. Averaging Formula for Nielsen numbers

Now we compare the Nielsen number $N(f)$ of f with the Nielsen numbers $N(\bar{\alpha}\bar{f})$ of the liftings $\bar{\alpha}\bar{f}$, $(\bar{\alpha} \in \pi/K)$, of f.

Using the above one-to-one correspondence we can prove the following.

Theorem 2.1. *Let X be a compact connected space and $f : X \longrightarrow X$ be a self-map. Then*

$$N(f) \geq \frac{1}{|\pi : K|} \sum_{\bar{\alpha} \in \pi/K} N(\bar{\alpha}\bar{f}),$$

and equality occurs if and only if for each $k \in K$ and $\alpha \in \pi$, the fixed point class $p\,\mathrm{Fix}(k\alpha\tilde{f})$ of f is inessential or $|q_{k\alpha}\,\mathrm{fix}(\tau_{k\alpha}\varphi)| = 1$.

Corollary 2.2. *If π is a finite group, then*

$$N(f) = \frac{1}{|\pi|} \sum_{\alpha \in \pi} |\,\mathrm{fix}(\tau_\alpha\varphi)| \cdot N(\alpha\tilde{f}) \geq \frac{1}{|\pi|} \sum_{\alpha \in \pi} N(\alpha\tilde{f}),$$

and equality occurs if and only if for each $\alpha \in \pi$, $|\,\mathrm{fix}(\tau_\alpha\varphi)| = 1$.

Example 2.3. Let $f : \mathbb{R}P^2 \longrightarrow \mathbb{R}P^2$ be the identity map and $\tilde{f} : S^2 \longrightarrow S^2$ be the identity map, i.e., $f = \mathrm{id}_{\mathbb{R}P^2}$ and $\tilde{f} = \mathrm{id}_{S^2}$. Then we have a commutative diagram

$$\begin{array}{ccc} S^2 & \xrightarrow{\ \tilde{f}\ } & S^2 \\ \Big\downarrow & & \Big\downarrow \\ \mathbb{R}P^2 & \xrightarrow{\ f\ } & \mathbb{R}P^2 \end{array}$$

The group of deck transformations is $\Pi = \{1, \alpha\} \cong \mathbb{Z}_2$. The homomorphism $\varphi : \Pi \longrightarrow \Pi$ corresponding to $\tilde{f}$ is the identity map. In particular, $\mathrm{fix}(\varphi) = \Pi$.

Note that $\alpha\tilde{f} = \alpha$ is the antipodal map on S^2. This map has no fixed point; $\mathrm{Fix}(\alpha\tilde{f}) = \emptyset$; $p\,\mathrm{Fix}(\alpha\tilde{f})$ is an inessential fixed point class of f and $N(\alpha\tilde{f}) = 0$.

Next we compute $N(\tilde{f})$ and find out whether or not $p\,\mathrm{Fix}(\tilde{f})$ is an essential fixed point class of $\mathrm{Fix}(f)$. By Remark 1.4 with $K = \{1\}$, we have $L(\tilde{f}) = |\,\mathrm{fix}(\varphi)| \cdot \mathrm{ind}(f, p\,\mathrm{Fix}(\tilde{f}))$. Thus

$$2 = L(\tilde{f}) = |\,\mathrm{fix}(\varphi)| \cdot \mathrm{ind}(f, p\,\mathrm{Fix}(\tilde{f})) = 2 \cdot \mathrm{ind}(f, p\,\mathrm{Fix}(\tilde{f})).$$

Hence $p\,\mathrm{Fix}(\tilde{f})$ is an essential fixed point class of f and $N(\tilde{f}) = 1$.

In all, $p\,\mathrm{Fix}(\tilde{f})$ is an essential fixed point class and $p\,\mathrm{Fix}(\alpha\tilde{f})$ is an inessential fixed point class of f. Hence $N(f) = 1$, but $\frac{1}{|\Pi|}\sum_{\alpha \in \Pi} N(\alpha\tilde{f}) = \frac{1}{2}(N(\tilde{f}) + N(\alpha\tilde{f})) = \frac{1}{2}(1 + 0) = \frac{1}{2}$. This shows that $N(f) \neq \frac{1}{|\Pi|}\sum_{\alpha \in \Pi} N(\alpha\tilde{f})$.

Using [4, Theorem1.1] and [1], we prove the averaging formula for Nielsen numbers for continuous maps on infra-nilmanifolds.

Theorem 2.4. *Let $f : M \longrightarrow M$ be any continuous map on an infra-nilmanifold M. Let M_K be a regular covering of M which is a compact nilmanifold with $\pi_1(M_K) = K$. Assume $f_*(K) \subset K$. Then*

$$N(f) = \frac{1}{[\pi : K]} \sum N(\bar{f}),$$

where the sum ranges over all the liftings $\bar{f}$ of f. In particular, $N(f) \geq |L(f)|$.

Remark 2.5. In Theorem 2.4, we are assuming that $f : M \longrightarrow M$ and K is a finite-index, normal subgroup of the pure translations Γ of the infra-nilmanifold M so that $f_*(K) \subset K$. However it is known that such a K always exists. In fact, it is shown in [2, Lemma 1.1] that there exists a fully invariant subgroup $K \subset \Gamma$ of $\pi_1(M)$ which is of finite index. Therefore for any $f : M \longrightarrow M$, $f_*(K) \subset K$.

3. Mod K Nielsen numbers

It is clear from the definition of mod K fixed point class that if every essential mod K fixed point class coincides with an ordinary fixed point class, then $N_K(f) = N(f)$. We show that the converse is also true for every continuous maps on infra-nilmanifolds.

Theorem 3.1. *Let $f : M \longrightarrow M$ be any continuous map on an infra-nilmanifold M. Let M_K be a regular covering of M which is a compact nilmanifold with $\pi_1(M_K) = K$. Assume $f_*(K) \subset K$. Then*

$$N(f) \geq N_K(f),$$

and equality occurs if and only if every essential mod K fixed point class coincides with an ordinary fixed point class of f.

The next proposition tells us how to compute $|\mathrm{im}(\hat{j}_\alpha)|$.

Proposition 3.2. *Let $f : M \longrightarrow M$ be any continuous map on an infra-nilmanifold M. Suppose M_K is a regular covering of M which is a compact nilmanifold with $\pi(M_K) = K$. Assume that $f_*(K) \subset K$. Then every essential mod K fixed point class $\bar{p}\,\mathrm{Fix}(\bar{\alpha}\bar{f})$ contains exactly $|\mathrm{im}(\hat{j}_\alpha)|$ ordinary fixed point classes of f, where*

$$|\mathrm{im}(\hat{j}_\alpha)| = \frac{N(\bar{\alpha}\bar{f})}{|\,\mathrm{fix}(\tau_{\bar{\alpha}}\bar{\varphi})|} = \frac{N(\bar{\alpha}\bar{f})}{[\Pi : K]} \cdot |[\bar{\alpha}]|.$$

Example 3.3. Let G be the 3-dimensional Heisenberg group. That is,

$$G = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}.$$

We denote this general element by $\{x, y, z\}$. Let Γ be the subgroup of G which is generated by

$$\{0, 0, 1\}, \quad \{0, 1, 0\}, \quad \{1, 0, 0\}.$$

Then Γ is the subgroup of G consisting of all integral matrices, and hence is a lattice of G.

Let $a = \{0, 0, \frac{1}{2}\} \in G$ and $A : G \to G$ be the automorphism of G given by

$$A(\{x, y, z\}) = \{-x, -y, z\}.$$

Then A has period 2, and $(a, A)^2 = (a^2, I) = (\{0, 0, 1\}, I) \in G \rtimes \mathrm{Aut}(G)$, where I is the identity automorphism of G. The subgroup

$$\Pi = \langle \Gamma, (a, A) \rangle \subset G \rtimes \mathrm{Aut}(G)$$

generated by the lattice Γ and the element (a, A) is discrete and torsion free, and Γ is a normal subgroup of Π of index 2. Thus Π is an almost Bieberbach group, and $\Pi\backslash G$ is an infra-nilmanifold, which has a double covering $\Gamma\backslash G \longrightarrow \Pi\backslash G$ by its holonomy group, $\Psi = \Pi/\Gamma = \{1, A\} \cong \mathbb{Z}_2$.

Let $D : G \longrightarrow G$ be the automorphism of G given by

$$D(\{x,y,z\}) = \{x+y, x, -z + \frac{1}{2}x^2 + xy\}.$$

Then $DA = AD$ and the conjugation by $(\{0,0,0\}, D) \in G \rtimes \mathrm{Aut}(G)$ maps Π into Π (and Γ into Γ). Thus, the affine map $(\{0,0,0\}, D) : G \to G$ induces $\phi_D : \Gamma\backslash G \to \Gamma\backslash G$ and $\Phi_D : \Pi\backslash G \to \Pi\backslash G$ so that the following diagram is commutative:

$$\begin{array}{ccc} G & \xrightarrow{(\{0,0,0\},D)} & G \\ \downarrow & & \downarrow \\ \Gamma\backslash G & \xrightarrow{\phi_D} & \Gamma\backslash G \\ \downarrow & & \downarrow \\ \Pi\backslash G & \xrightarrow{\Phi_D} & \Pi\backslash G \end{array}$$

We take an ordered (linear) basis for the Lie algebra $\mathfrak{G}$ of G as follows:

$$\mathbf{e}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

With respect to this basis, the differentials of A and D are

$$A_* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad D_* = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Therefore by Theorem 2.4, the Nielsen number of the map $\Phi_D : \Pi\backslash G \longrightarrow \Pi\backslash G$ is:

$$\begin{aligned} N(\Phi_D) &= \frac{1}{|\Psi|} \sum_{\psi \in \Psi} N(\psi\phi_D) = \frac{1}{|\Psi|} \sum_{\psi \in \Psi} |\det(I - \psi_* D_*)| \\ &= \frac{1}{2}(|\det(I - D_*)| + |\det(I - A_* D_*)|) \\ &= \frac{1}{2}(|-2| + |2|) = 2. \end{aligned}$$

Let $\alpha = (a, A)$. Since $(\{0,0,0\}, D)\alpha = (D(a)\cdot a^{-1}, I)\alpha(\{0,0,0\}, D)$ and $D(a)\cdot a^{-1} \in \Gamma$, we have $\phi_D \bar{\alpha} = \bar{\alpha}\phi_D$. This means that $\bar{\varphi} : \Pi/\Gamma \longrightarrow \Pi/\Gamma$ maps $\bar{\alpha}$ to $\bar{\alpha}$. Hence $\bar{\varphi}$ is the identity map, and the Reidemeister classes of $\bar{\varphi}$ are singleton sets.

By Proposition 3.2,

$$\begin{aligned} |\mathrm{im}(\hat{i}_{\{0,0,0\}})| &= \frac{N(\phi_D)}{[\Pi : \Gamma]} \cdot |[\bar{1}]| = \frac{|\det(I - D_*)|}{2} \cdot 1 = \frac{|-2|}{2} \cdot 1 = 1, \\ |\mathrm{im}(\hat{j}_\alpha)| &= \frac{N(\bar{\alpha}\phi_D)}{[\Pi : \Gamma]} \cdot |[\bar{\alpha}]| = \frac{|\det(I - A_* D_*)|}{2} \cdot 1 = \frac{|2|}{2} \cdot 1 = 1. \end{aligned}$$

Every mod Γ fixed point classes contains only one ordinary fixed point class. By Corollary 3.1,

$$N_\Gamma(\Phi_D) = N(\Phi_D) = 2.$$

Example 3.4. Let G be the 3-dimensional Heisenberg group. Let K be the subgroup of G consisting of all integral matrices of G. Let $a = \{0, 0, \frac{1}{4}\}$ and $A : G \to G$ be the automorphism of G given by

$$A(\{x, y, z\}) = \{-x, -y, z\}.$$

Then A has period 2, and $(a, A)^4 = (a^4, I) = (\{0, 0, 1\}, I) \in G \rtimes \mathrm{Aut}(G)$, where I is the identity automorphism of G. The subgroup

$$\Pi = \langle K, (a, A)\rangle \subset G \rtimes \mathrm{Aut}(G)$$

generated by the lattice K and the element (a, A) is discrete and torsion free, and K is a normal subgroup of Π of index 4. Note that $\Gamma = \Pi \cap G$ contains $(a, A)^2 = (a^2, I)$ so that K has index 2 in Γ.

Let $D \in \mathrm{Aut}(G)$ be given by

$$D(\{x, y, z\}) = \{3y, -x, 3z - 3xy\}.$$

Then $(\{0, 0, 0\}, D)(a, A)(\{0, 0, 0\}, D)^{-1} = (a, A)^3$, and the conjugation by $(\{0, 0, 0\}, D)$ maps Π into Π, and K into K. Therefore, the affine map $(\{0, 0, 0\}, D) : G \to G$ induces $\phi_D : K\backslash G \to K\backslash G$ and $\Phi_D : \Pi\backslash G \to \Pi\backslash G$.

With respect to the standard linear basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ for the Lie algebra $\mathfrak{G}$ of G, the differentials of A and D are

$$A_* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad D_* = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & -1 & 0 \end{bmatrix}.$$

Let $\alpha = (a, A) \in \Pi$. Then Π/K is the cyclic group of order 4 generated by $\bar{\alpha}$. The Nielsen number of the map $\Phi_D : \Pi\backslash G \longrightarrow \Pi\backslash G$ is, by Theorem 2.4,:

$$\begin{aligned} N(\Phi_D) =& \frac{1}{[\Pi : K]} \sum_{\psi \in \Pi/K} N(\psi\phi_D) \\ =& \frac{1}{4}(N(\phi_D) + N(\bar{\alpha}\phi_D) + N(\bar{\alpha}^2\phi_D) + N(\bar{\alpha}^3\phi_D)) \\ =& \frac{1}{4}(|\det(I - D_*)| + |\det(I - A_*D_*)| \\ &+ |\det(I - D_*)| + |\det(I - A_*D_*)|) \\ =& \frac{1}{4}(|-8| + |-8| + |-8| + |-8|) = 8. \end{aligned}$$

Observe that the homomorphism $\bar{\varphi} : \Pi/K \longrightarrow \Pi/K$ maps $\bar{\alpha}$ to $\bar{\alpha}^{-1}$. Its Reidemeister classes are $[\bar{1}] = [\bar{\alpha}^2] = \{1, \bar{\alpha}^2\}$ and $[\bar{\alpha}] = [\bar{\alpha}^{-1}] = \{\bar{\alpha}, \bar{\alpha}^{-1}\}$. By Proposition 3.2, $|\mathrm{im}(\hat{i}_\alpha)| = N(\bar{\alpha}\phi_D)/[\Pi : K] \cdot |[\bar{\alpha}]| = \frac{|-8|}{4} \cdot 2 = 4$. Every mod K fixed point class consists of 4 ordinary fixed point classes. Hence

$$N_K(\Phi_D) = 2 < 8 = N(\Phi_D).$$

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