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THE HEAT ENERGY CONTENT OF A RIEMANNIAN MANIFOLD

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ABSTRACT. We survey some recent results concerning the heat energy content of a Riemannian manifold with both local and non-local boundary conditions

Let M be a compact Riemannian manifold with smooth boundary ∂M. Let D be an operator of Laplace type acting on the space of smooth sections to a vector bundle V over M; for example, one could take $D = d^*d$ to be the scalar Laplacian. Let ϕ be the initial temperature of the manifold. Let $\mathcal{B}$ be a suitable boundary condition. The subsequent temperature u of the manifold is governed by the equations:

$$(\partial_t + D)u = 0,\ u(x;0) = \phi(x),\ \mathcal{B}u = 0.$$

Let ρ be the specific heat of the manifold. The total heat energy content of the manifold is then given by

$$\beta(\phi, \rho, D, \mathcal{B})(t) = \textstyle\int_M u(x;t)\rho(x).$$

There is an asymptotic expansion valid for short time of the form:

$$\beta(\phi, \rho, D, \mathcal{B})(t) \sim \textstyle\sum_{n\geq 0} \beta_n(\phi, \rho, D, \mathcal{B})t^{n/2}.$$

These coefficients will be the focus of our study. We adopt the Einstein convention and sum over repeated indices. If we expand D locally in the form:

$$D = -(g^{\mu\nu}\partial_\mu\partial_\nu + A^\mu\partial_\mu + B),$$

then the leading symbol $g^{\mu\nu}$ describes the inverse of the Riemannian metric on the manifold and is invariantly defined. The remaining terms A^μ and B are not, however, invariantly defined. Let Γ be the Christoffel symbols of the Levi-Civita connection. There is an invariantly defined connection ∇ with connection 1 form ω on V and an endomorphism E of V whose equations of structure are defined by:

$$\omega_\delta := \tfrac{1}{2}g_{\nu\delta}(a^\nu + g^{\mu\sigma}\Gamma_{\mu\sigma}{}^\nu) \text{ and } E := B - g^{\nu\mu}(\partial_\nu\omega_\mu + \omega_\nu\omega_\mu - \omega_\sigma\Gamma_{\nu\mu}{}^\sigma)$$

so that D can be written in the form:

$$Du = -(g^{\mu\nu}u_{;\mu\nu} + Eu).$$

It is in terms of these tensors that we will express the local invariants β_n.

Let indices $\{i, j, k, l\}$ range from 1 to $m := \dim(M)$ and index a local orthonormal frame $\{e_i\}$ for the tangent bundle TM. Near the boundary, we normalize the

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choice of frame so e_m is the inward unit geodesic normal. Let indices $\{a,b,c\}$ range from 1 to $m-1$ and index the induced orthonormal frame for $T(\partial M)$. Let R_{ijkl} be the components of the curvature tensor with the sign convention that the scalar curvature is given by $\tau := R_{ijji}$. Let L_{ab} be the components of the second fundamental form and let Ω be the curvature of the connection ∇ on V. It is convenient to regard the specific heat as a section to V^*; let $\langle\cdot,\cdot\rangle$ be the natural pairing between V and V^*; we shall occasionally omit these brackets in the interests of brevity. Let $\tilde{D}$ and $\tilde{\mathcal{B}}$ be the associated operators and boundary conditions on V^*. We integrate with respect to the Riemannian measures on M and on ∂M and refer to [1, 4, 7, 10, 16] for the proof of the following results:

Theorem 1. *Let $\mathcal{B}$ denote Dirichlet boundary conditions. Then:*
1) $\beta_0(\phi,\rho,D,\mathcal{B}) = \int_M \langle\phi,\rho\rangle$.
2) $\beta_1(\phi,\rho,D,\mathcal{B}) = -\frac{2}{\sqrt{\pi}}\int_{\partial M}\langle\phi,\rho\rangle$.
3) $\beta_2(\phi,\rho,D,\mathcal{B}) = -\int_M\langle D\phi,\rho\rangle + \int_{\partial M}\{\langle\frac{1}{2}L_{aa}\phi,\rho\rangle - \langle\phi,\rho_{;m}\rangle\}$.
4) $\beta_3(\phi,\rho,D,\mathcal{B}) = -\frac{2}{\sqrt{\pi}}\int_{\partial M}\{\frac{2}{3}\langle\phi_{;mm},\rho\rangle + \frac{2}{3}\langle\phi,\rho_{;mm}\rangle - \langle\phi_{:a},\rho_{:a}\rangle + \langle E\phi,\rho\rangle$
$-\frac{2}{3}L_{aa}\langle\phi_{;m},\rho\rangle - \frac{2}{3}L_{aa}\langle\phi,\rho_{;m}\rangle + \langle(\frac{1}{12}L_{aa}L_{bb} - \frac{1}{6}L_{ab}L_{ab} + \frac{1}{6}R_{amam})\phi,\rho\rangle\}$.
5) $\beta_4(\phi,\rho,D,\mathcal{B}) = \frac{1}{2}\int_M\langle D\phi,\tilde{D}\rho\rangle + \int_{\partial M}\{\frac{1}{2}\langle(D\phi)_{;m},\rho\rangle + \frac{1}{2}\langle\phi,(\tilde{D}\rho)_{;m}\rangle$
$-\frac{1}{4}\langle L_{aa}D\phi,\rho\rangle - \frac{1}{4}\langle L_{aa}\phi,\tilde{D}\rho\rangle + \langle(\frac{1}{8}E_{;m} - \frac{1}{16}L_{ab}L_{ab}L_{cc} + \frac{1}{8}L_{ab}L_{ac}L_{bc}$
$-\frac{1}{16}R_{ambm}L_{ab} + \frac{1}{16}R_{abcb}L_{ac} + \frac{1}{32}\tau_{;m} + \frac{1}{16}L_{ab:ab})\phi,\rho\rangle - \frac{1}{4}L_{ab}\langle\phi_{:a},\rho_{:b}\rangle$
$-\frac{1}{8}\langle\Omega_{am}\phi_{:a},\rho\rangle + \frac{1}{8}\langle\Omega_{am}\phi,\rho_{:a}\rangle\}$.

Theorem 2. *Let $\mathcal{B} = \nabla_m + S$ define Robin boundary conditions.*
1) $\beta_0(\phi,\rho,D,\mathcal{B}) = \int_M\langle\phi,\rho\rangle$.
2) $\beta_1(\phi,\rho,D,\mathcal{B}) = 0$.
3) $\beta_2(\phi,\rho,D,\mathcal{B}) = -\int_M\langle D\phi,\rho\rangle + \int_{\partial M}\langle\mathcal{B}\phi,\rho\rangle$.
4) $\beta_3(\phi,\rho,D,\mathcal{B}) = \frac{4}{3\sqrt{\pi}}\int_{\partial M}\langle\mathcal{B}\phi,\tilde{\mathcal{B}}\rho\rangle$.
5) $\beta_4(\phi,\rho,D,\mathcal{B}) = \frac{1}{2}\int_M\langle D\phi,\tilde{D}\rho\rangle + \int_{\partial M}\{-\frac{1}{2}\langle\mathcal{B}\phi,\tilde{D}\rho\rangle - \frac{1}{2}\langle D\phi,\tilde{\mathcal{B}}\rho\rangle$
$+\langle(\frac{1}{2}S + \frac{1}{4}L_{aa})\mathcal{B}\phi,\tilde{\mathcal{B}}\rho\rangle\}$.

Savo [17] has derived formulae for all the coefficients in the special case that D is the scalar Laplacian and $\rho=\phi=1$ reflect constant initial temperature and unit density. Fractal boundaries can also be considered [6, 8] as can inhomogeneous problems [2, 3].

Oblique boundary conditions can be studied. Let $\mathcal{B}_T$ be a tangential differential operator and let *oblique boundary conditions* be defined by the operator:

$$\mathcal{B}\psi := (\nabla_m + \mathcal{B}_T)\psi|_{\partial M}.$$

Express $\mathcal{B}_T = \Gamma_a\nabla_a + \nabla_a\Gamma_a + S$ with respect to suitably chosen auxiliary endomorphisms Γ and S. We refer to [13] for the proof of the following result:

Theorem 3. *Adopt the notation established above*
1) $\beta_0(\phi,\rho,D,\mathcal{B}) = \int_M\langle\phi,\rho\rangle$.
2) $\beta_1(\phi,\rho,D,\mathcal{B}) = 0$.
3) $\beta_2(\phi,\rho,D,\mathcal{B}) = -\int_M\langle D\phi,\rho\rangle + \int_{\partial M}\langle\mathcal{B}\phi\rho\rangle$.
4) $\beta_3(\phi,\rho,D,\mathcal{B}) = \frac{4}{3\sqrt{\pi}}\int_{\partial M}\langle\mathcal{B}\phi,\tilde{\mathcal{B}}\rho\rangle$.
5) $\beta_4(\phi,\rho,D,\mathcal{B}) = \frac{1}{2}\int_M\langle D\phi,\tilde{D}\rho\rangle + \int_{\partial M}\{-\frac{1}{2}\langle\mathcal{B}\phi,\tilde{D}\rho\rangle - \frac{1}{2}\langle D\phi,\tilde{\mathcal{B}}\rho\rangle$
$+\langle(\frac{1}{2}\mathcal{B}_T + \frac{1}{4}L_{aa})\mathcal{B}\phi,\tilde{\mathcal{B}}\rho\rangle\}$.

Heat transfer and transmittal boundary conditions appear in many physical settings [9]. Let $(M_\pm, g_\pm)$ be smooth compact m dimensional Riemannian manifolds. We assume that $\Sigma = \partial M_+ = \partial M_-$ is a smooth $m-1$ dimensional manifold and that $g_+|_\Sigma = g_-|_\Sigma$. Let $D_\pm$ be operators of Laplace type on vector bundles $V_\pm$ over $M_\pm$. Let $\nu_\pm$ be the inward unit normals of $\Sigma \subset M_\pm$; note that $\nu_+ = -\nu_-$. Let $\phi := (\phi_+, \phi_-)$ and $\rho := (\rho_+, \rho_-)$.

Suppose that $V_+|_\Sigma = V_-|_\Sigma$ and that there is given an auxiliary endomorphism U of $V_\Sigma := V_\pm|_\Sigma$ serving as an impedance matching term. Let $\nabla^\pm$ be the natural connections defined by the operators $D_\pm$. Let

$$\mathcal{B}\phi := \{\phi_+|_\Sigma - \phi_-|_\Sigma\} \oplus \{(\nabla^+_{\nu_+}\phi_+)|_\Sigma + (\nabla^-_{\nu_-}\phi_-)|_\Sigma - U\phi_+|_\Sigma\};$$

ϕ satisfies these boundary conditions if and only if ϕ extends continuously across the interface Σ and the normal derivatives match, modulo the impedance matching term U. We refer to [12] for the proof of the following result; we suppress the formula for β_3 in the interests of brevity:

Theorem 4. *Let $\mathcal{B}$ be the boundary conditions defined above.*

1) $\beta_0(\phi,\rho,D,\mathcal{B}) = \int_{M_+}\phi_+\rho_+ + \int_{M_-}\phi_-\rho_-$.

2) $\beta_1(\phi,\rho,D,\mathcal{B}) = \int_\Sigma\{-\frac{1}{\sqrt{\pi}}(\phi_+\rho_+ + \phi_-\rho_-) + \frac{1}{\sqrt{\pi}}(\phi_+\rho_- + \phi_-\rho_+)\}$.

3) $\beta_2(\phi,\rho,D,\mathcal{B}) = -\int_{M_+} D_+\phi_+\cdot\rho_+ - \int_{M_-} D_-\phi_-\cdot\rho_-$
$+\int_\Sigma\big\{\frac{1}{8}(\phi_+\rho_+L^+_{aa} + \phi_-\rho_-L^-_{bb}) + \frac{1}{8}(\phi_+\rho_+L^-_{aa} + \phi_-\rho_-L^+_{aa})$
$-\frac{1}{8}(\phi_+\rho_-L^+_{aa} + \phi_-\rho_+L^-_{aa}) - \frac{1}{8}(\phi_+\rho_-L^-_{aa} + \phi_-\rho_+L^+_{aa})$
$+\frac{1}{2}(\phi_{+;\nu_+}\rho_+ + \phi_{-;\nu_-}\rho_-) + \frac{1}{2}(\phi_{+;\nu_+}\rho_- + \phi_{-;\nu_-}\rho_+)$
$-\frac{1}{2}(\phi_+\rho_{+;\nu_+} + \phi_-\rho_{-;\nu_-}) + \frac{1}{2}(\phi_+\rho_{-;\nu_-} + \phi_-\rho_{+;\nu_+})$
$-\frac{1}{4}(\phi_+\rho_+ + \phi_-\rho_-)U - \frac{1}{4}(\phi_+\rho_- + \phi_-\rho_+)U\big\}$.

One can also study boundary conditions of the form

$$\mathcal{B}\phi := \left\{\begin{pmatrix} \nabla^+_{\nu_+} + S_{++} & S_{+-} \\ S_{-+} & \nabla^-_{\nu_-} + S_{--}\end{pmatrix}\begin{pmatrix}\phi_+ \\ \phi_-\end{pmatrix}\right\}\Bigg|_\Sigma \tag{0.1}$$

where $S_{\varepsilon\delta} : V_\delta|_\Sigma \to V_\varepsilon|_\Sigma$. We refer to [12] for the proof of the following result:

Theorem 5. 1) $\beta_0(\phi,\rho,D,\mathcal{B}) = \int_M \phi\rho$.

2) $\beta_1(\phi,\rho,D,\mathcal{B}) = 0$.

3) $\beta_2(\phi,\rho,D,\mathcal{B}) = -\int_M D\phi\cdot\rho + \int_\Sigma \mathcal{B}\phi\cdot\rho$.

4) $\beta_3(\phi,\rho,D,\mathcal{B}) = \frac{4}{3\sqrt{\pi}}\int_\Sigma \mathcal{B}\phi\cdot\tilde{\mathcal{B}}\rho$.

Time dependent geometries play an important role in many investigations. Let D_t be a smooth 1 parameter family of operators of Laplace type on the space of smooth sections to a vector bundle V over a compact Riemannian manifold M. We assume the leading symbol is given by the metric g_t; for example, we could take $D_t = \Delta_t$ to be the scalar Laplacian on the trivial vector bundle defined by the metric g_t; this is the canonical example in this subject. We expand

$$D_t := D_0 + \textstyle\sum_{r>0} t^r\{\mathcal{G}_{r,ij}(x)\nabla_i\nabla_j + \mathcal{F}_{r,i}(x)\nabla_i + \mathcal{E}_r(x)\}.$$

Let $p(x;t)$ be a time dependent heat source on M.

To discuss Dirichlet and Neumann boundary conditions simultaneously, we suppose given decomposition $\partial M = C_D \dot\sqcup D_N$ as a disjoint union. We use the following operator to define our boundary conditions:

$$\mathcal{B}u := u|_{C_D} \oplus (u_{;m} + Su)|_{C_N}.$$

Here $u_{;m}$ is the inward unit normal covariant derivative of u and S is an auxiliary endomorphism of V restricted to C_N. The boundary operator is static - i.e. independent of t. Let $\psi(y;t)$ be a smooth section to V defined over ∂M. On the Neumann boundary component C_N, we use a Neumann heat pump - i.e. we pump heat into M at a rate defined by ψ; the parameter S controls the coupling between the heat transfer and the temperature difference on the Neumann component. On the Dirichlet component we use a Dirichlet heat pump keep the temperature at ψ. The temperature distribution $u_{p,\phi,\psi;D}(x;t)$ which is defined by these data is the solution to the equations

$$(\partial_t + D_t)u_{p,\phi,\psi;D}(x;t) = p(x;t),\ u_{p,\phi,\psi;D}(x;0) = \phi(x), \text{ and } \mathcal{B}u = \psi. \tag{0.2}$$

We refer to [5, 11, 15] for the proof of the following result:

Theorem 6.

(1) $\beta_0(p,\phi,\psi,\rho;D) = \int_M \langle \phi, \rho\rangle$.

(2) $\beta_1(p,\phi,\psi,\rho;D) = -\frac{2}{\sqrt{\pi}}\int_{C_D}\{\langle \phi - \psi_0^{\mathcal{D}}, \rho\rangle\}$.

(3) $\beta_2(p,\phi,\psi,\rho;D) = -\int_M\{\langle D_0\phi,\rho\rangle - \langle p_0,\rho\rangle\} + \int_{C_N}\{\langle(\mathcal{B}\phi - \psi_0^{\mathcal{N}}),\rho\rangle\}$
$+\int_{C_D}\{\langle\frac{1}{2}L_{aa}(\phi-\psi_0^{\mathcal{D}}),\rho\rangle - \langle(\phi-\psi_0^{\mathcal{D}}),\rho_{;m}\rangle\}$.

(4) $\beta_3(p,\phi,\psi,\rho;D) = -\frac{2}{\sqrt{\pi}}\int_{C_D}\{\frac{2}{3}\langle p_0,\rho\rangle - \frac{2}{3}\langle D_0\phi,\rho\rangle - \frac{2}{3}\langle(\phi-\psi_0^{\mathcal{D}}),\tilde{D}_0\rho\rangle$
$+\frac{1}{3}\langle(\phi-\psi_0^{\mathcal{D}})_{:a},\rho_{:a}\rangle - \frac{2}{3}\langle\psi_1^{\mathcal{D}},\rho\rangle + \langle(-\frac{1}{3}E + \frac{1}{12}L_{aa}L_{bb}$
$-\frac{1}{6}L_{ab}L_{ab} + \frac{1}{6}R_{amam} - \mathcal{G}_{1,mm})(\phi-\psi_0^{\mathcal{D}}),\rho\rangle\}$
$+\frac{4}{3\sqrt{\pi}}\int_{C_N}\{\langle(\mathcal{B}\phi-\psi_0^{\mathcal{N}}),\tilde{\mathcal{B}}\rho\rangle\}$.

(5) $\beta_4(p,\phi,\psi,\rho;D) = \frac{1}{2}\int_M\{\langle p_1,\rho\rangle - \langle D_0p_0,\rho\rangle + \langle D_0\phi,\tilde{D}_0\rho\rangle$
$-\langle(\mathcal{G}_{1,ij}\phi_{;ij} + \mathcal{F}_{1,i}\phi_{;i} + \mathcal{E}_1\phi),\rho\rangle\}$.
$+\int_{C_D}\{\frac{1}{4}L_{aa}\langle p_0,\rho\rangle - \frac{1}{2}\langle p_0,\rho_{;m}\rangle - \frac{1}{4}L_{aa}\langle\psi_1^{\mathcal{D}},\rho\rangle + \frac{1}{2}\langle\psi_1^{\mathcal{D}},\rho_{;m}\rangle$
$+\frac{1}{2}\langle(D_0\phi)_{;m},\rho\rangle + \frac{1}{2}\langle(\phi-\psi_0^{\mathcal{D}}),(\tilde{D}_0\rho)_{;m}\rangle - \frac{1}{4}\langle L_{aa}D_0\phi,\rho\rangle$
$-\frac{1}{4}\langle L_{aa}(\phi-\psi_0^{\mathcal{D}}),\tilde{D}_0\rho\rangle + \langle(\frac{1}{8}E_{;m} - \frac{1}{16}L_{ab}L_{ab}L_{cc} + \frac{1}{8}L_{ab}L_{ac}L_{bc}$
$-\frac{1}{16}R_{ambm}L_{ab} + \frac{1}{16}R_{abcb}L_{ac} + \frac{1}{32}\mathcal{R}_{;m} + \frac{1}{16}L_{ab:ab})(\phi-\psi_0^{\mathcal{D}}),\rho\rangle$
$-\frac{1}{4}L_{ab}\langle(\phi-\psi_0^{\mathcal{D}})_{:a},\rho_{:b}\rangle - \frac{1}{8}\langle\Omega_{am}(\phi-\psi_0^{\mathcal{D}})_{:a},\rho\rangle + \frac{1}{8}\langle\Omega_{am}(\phi-\psi_0^{\mathcal{D}}),\rho_{:a}\rangle$
$+(\frac{7}{16}\mathcal{G}_{1,mm;m} - \frac{1}{4}\mathcal{G}_{1,mm}L_{aa} - \frac{5}{16}\mathcal{F}_{1,m})\langle(\phi-\psi_0^{\mathcal{D}}),\rho\rangle$
$-\frac{5}{16}\mathcal{G}_{1,am}\langle(\phi-\psi_0^{\mathcal{D}})_{:a},\rho\rangle + \frac{1}{2}\mathcal{G}_{1,mm}\langle(\phi-\psi_0^{\mathcal{D}}),\rho_{;m}\rangle\}$
$+\int_{C_N}\{\frac{1}{2}\langle\mathcal{B}p_0,\rho\rangle - \frac{1}{2}\langle(\mathcal{B}\phi-\psi_0^{\mathcal{N}}),\tilde{D}_0\rho\rangle - \frac{1}{2}\langle D_0\phi,\tilde{\mathcal{B}}\rho\rangle - \frac{1}{2}\langle\psi_1^{\mathcal{N}},\rho\rangle$
$+\langle(\frac{1}{2}S + \frac{1}{4}L_{aa})(\mathcal{B}\phi-\psi_0^{\mathcal{N}}),\tilde{\mathcal{B}}\rho\rangle - \frac{1}{2}\mathcal{G}_{1,mm}\langle(\mathcal{B}\phi-\psi_0^{\mathcal{N}}),\rho\rangle\}$.

Spectral boundary conditions can be treated as well. Let $P = \gamma^\nu\nabla_\nu + \psi$ be an operator of Dirac type and let $D := P^2$ be the associated operator of Laplace type. Decompose $P = \gamma_m(\nabla_m + B)$ and let $A := \frac{1}{2}(B + B^*) + \Theta$ where the adjoint B^* is defined with respect to the structures on ∂M and Θ is self-adjoint. Let orthogonal projection Π on the positive spectrum of A; to simplify matters, assume $\ker(A) = \{0\}$. Let $D = P^2$ and let $D_{\mathcal{B}}$ is the realization of D defined by the operator

$$\mathcal{B}\psi := \Pi\psi|_{\partial\mathcal{M}} \oplus \Pi P\psi|_{\partial\mathcal{M}}.$$

We shall also assume that $\gamma_m A = -A\gamma_m$ so $\gamma_m\Pi = \{\mathrm{Id} - \Pi\}\gamma_m$ as this is the case in most applications. The operators P_Π and $D_{\mathcal{B}} = P_\Pi^2$ are then self-adjoint and the following formula is believed to hold [14]:

Theorem 7. (1) $\beta_0(\phi,\rho,D,\mathcal{B}) = \int_M \langle\phi,\rho\rangle$.
(2) $\beta_1(\phi,\rho,D,\mathcal{B}) = 2\pi^{-1/2}\int_{\partial M} -\langle\Pi\phi,\Pi\rho\rangle$.
(3) $\beta_2(\phi,\rho,D,\mathcal{B}) = -\int_M \langle P\phi,P\rho\rangle + \int_{\partial M}\{\langle\Pi\gamma_m P\phi,\Pi\rho\rangle + \langle\Pi\phi,\Pi\gamma_m P\rho\rangle + \frac{1}{2}L_{aa}\langle\Pi\phi,\Pi\rho\rangle$.

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