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# VOLUME AND PROJECTIVE CHANGE OF METRICS

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ABSTRACT. We study developments of projective transformations and present nonexistence of a nontrivial pointwise projective transformation from a noncompact complete Riemannian manifold $(M, g)$ to $(M, \overline{g})$ with conditions on volume growth and scalar curvature of $(M, g)$, and Ricci curvature of $(M, \overline{g})$.

## 1. Introduction

In this manuscript, we study Riemannian spaces which admit same geodesic with possibly different parameterizations. In physics, a geodesic represents the equation of motion, which describes all the phenomena. These space provide two ambient spaces which produce same physical phenomena. The Riemannian metric is determined by the angle and distance. The distance is measured along a geodesic. A geodesic is the most important concept of Riemannian geometry, which is used for calculating angle, distance, and other invariants of a manifold. Two of main tools for the study of Riemannian metrics are a conformal transformation and a projective transformation which preserve the angle and the geodesic, respectively. It was known that a given Riemannian metric is determined by conformal structure and projective structure (see Weyl [W]).

Here we study developments of the projective transformation and sketch the nonexistence of a nontrivial pointwise projective transformation from a noncompact complete Riemannian manifold $(M, g)$ to $(M, \overline{g})$ with conditions on volume growth and scalar curvature of $(M, g)$ and Ricci curvature of $(M, \overline{g})$.

## 2. Projective Transformations

A diffeomorphism $f$ from a Riemannian space $(M, g)$ onto a Riemannian space $(\overline{M}, \overline{g})$ is called a projective mapping if the image of every geodesic of $(M, g)$ is a geodesic in $(\overline{M}, \overline{g})$. A projective transformation is a projective mapping of a manifold onto itself. When a projective transformation $f : (M, g) \to (M, \overline{g})$ is the identity map, $f$ is called a pointwise projective transformation and two Riemannian metrics $g$ and $\overline{g}$ on a manifold $M$ are said to be pointwise projectively related. In this case, two manifolds have the same geodesics as point sets.

Let us consider a pair of projectively related metrics. Let $\Omega$ be a convex bounded open domain with smooth boundary $\partial\Omega$ in $R^n$. For $p, q \in \Omega$, define $d(p, q) = \ln \frac{|z-p|}{|z-q|}$ where $z$ denote the intersection point of the ray from $p$ to $q$. Note that $d(p, q) \neq d(q, p)$. This is called Func metric, which is a Finsler metric. By symmetrizing Func

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metric, we obtain Klein metric $\overline{d}(p,q) = \frac{1}{2}(d(p,q) + d(q,p))$, which is Riemannian. With some calculations, we see that geodesics on $\Omega$ with Klein metric are straight lines in $\Omega$. Therefore the Euclidean metric and Klein metric on $\Omega$ are projectively related. Note that Klein metric and Func metric are complete metrics on $\Omega$ with constant curvature $K = -1$ and $K = -\frac{1}{4}$, respectively (in Finsler sense).

We look over the developments of this subject. For two Riemannian metrics $g, \overline{g}$ on $M$, if they are projectively related and one of metric is a constant curvature metric then the other should be a constant curvature metric. This was proved by Beltrami using Ricci identity (see [Ei]). Later Hilbert proposed a problem related to projective transformation on Finsler manifolds.

**Hilbert's Fourth Problem:** Given a domain $\Omega \subset R^n$, determine all Finsler metrics on $\Omega$ whose geodesics are straight line.

Recent developments of this problem can be found in [Sz].

There is a way to measure how to check two metrics are not projectively related. The Weyl projective tensor is defined as

$$W^i_{jkl} = R^i_{ijk} - \frac{1}{n-1}(R_{jk}\delta^i_l - R_{jl}\delta^i_k).$$

It is known that if two metrics $g$ and $\overline{g}$ are projectively related, then corresponding Weyl tensors coincide. Moreover, if Weyl projective tensor of a metric vanishes on a domain, then this metric is a constant curvature metric on this domain(see [W]). The rigidity of projective transformation on constant curvature metric holds for Einstein metrics.

**Theorem 1** (Mike's)**.** *Let $(M, g)$ be a Riemannian n-manifold. Assume that $\overline{g}$ is another Riemannian metric pointwise projective to $g$. Suppose that $g$ is Einstein, then $\overline{g}$ must be Einstein metric.*

Previous studies on the nonexistence of projective mappings were done on a manifold with harmonic curvature [T, MR]. Manifolds admitting an infinitesimal projective transformation was studied on parallel Ricci space and constant scalar curvature space [Y, Y1]. Mikeš gives an example of nontrivial projective transformation [Mi]. However, it is hard to show the existence of a projective transformation on a general manifold. Harél [Ha] showed that there is a volume decreasing property on the projective transformation from a manifold with Ricci curvature bound below to a manifold with Ricci bounded above by negative constant.

**Theorem 2** (Harél)**.** *Let $f : M \to \overline{M}$ be a projective mapping of n-Riemannian manifold, $M$ being complete. If the Ricci curvature of $M$ is bounded below by a constant $-A$, and the Ricci curvature of $\overline{M}$ is bounded above by a constant $-b < 0$, then either $f$ is totally degenerate, or $A > 0$ and $f$ is volume decreasing up to a constant $(A/B)^{n/2}$.*

On Finsler case, Shen [Sh] recently showed that there exists no nontrivial pointwise projective transformation between Einstein metrics with other conditions. For the proof, Shen used a ordinary differential equation along a geodesic. For other developments of the projective transformation, we refer to Mikeš [Mi1].

## 3. Changes of metrics

Let $g$ and $\overline{g}$ be two Riemannian metrics on a manifold $M$. A necessary and sufficient condition for a mapping $f$ from $(M, g)$ to $(M, \overline{g})$ to be a pointwise projective transformation is that there exists a covector $\psi_j(x)$ satisfying the following

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_i \psi_j + \delta^h_j \psi_i, \tag{1}$$

in a common coordinate system $x$, where $\overline{\Gamma}$ ($\Gamma$) is the connection of $(M, \overline{g})$ ($(M, g)$), respectively (see [Sp] Vol. 2). In case the covector $\psi_j(x) \neq 0$, projective transformation is called nontrivial. Contracting (1) for $h$ and $j$, we have

$$2(n+1)\psi_i = \frac{\partial}{\partial x_i} \log \frac{\overline{G}}{G}, \tag{2}$$

where $\overline{G} = \det(\overline{g}_{ij})$ and $G = \det(g_{ij})$. Since $\frac{\overline{G}}{G}$ is an invariant, $\psi_i$ is the gradient of function $\psi$ where

$$2(n+1)\psi = \log \frac{\overline{G}}{G} + c_0, \tag{3}$$

for some constant $c_0$. Let $\psi_{,ij}$ be the second covariant derivative of $\psi$ with respect to the metric $g_{ij}$, in other words,

$$\psi_{,ij} = \frac{\partial^2 \psi}{\partial x_i \partial x_j} - \frac{\partial \psi}{\partial x^k} \Gamma^k_{ij} \tag{4}$$

and

$$\psi_{ij} = \psi_{,ij} - \psi_i \psi_j. \tag{5}$$

Curvature and Ricci curvature of $(M, g)$ are given by

$$R^m_{ijl} = \partial_j \Gamma^m_{il} - \partial_l \Gamma^m_{ij} + \Gamma^t_{il}\Gamma^m_{tj} - \Gamma^t_{ij}\Gamma^m_{tl}, \tag{6}$$

$$\begin{aligned} R_{ij} &= -R^l_{ijl} \\ &= \partial_l \Gamma^l_{ij} - \partial_j \Gamma^l_{il} - \Gamma^t_{il}\Gamma^l_{tj} + \Gamma^t_{ij}\Gamma^l_{tl}. \end{aligned} \tag{7}$$

Curvature formula and (1) implies

$$\overline{R^m_{ijl}} = R^m_{ijl} + \delta^m_l \psi_{ij} - \delta^m_j \psi_{il}. \tag{8}$$

Contracting (8) with $m$ and $l$, we have

$$\overline{R}_{ij} = R_{ij} - (n-1)\psi_{ij}. \tag{9}$$

We denote $S = g^{ij} R_{ij}$, scalar curvature of $(M, g)$, $\overline{R_{ij}}$, Ricci curvature of $(M, \overline{g})$, $f^+(x) = \max(f(x), 0)$, and $f^-(x) = \min(f(x), 0)$. By taking trace of (9), we have:

**Theorem 3.** *Let $(M, g)$ and $(M, \overline{g})$ be compact Riemannian manifolds. Assume that $(M, g)$ has nonnegative total scalar curvature and $(M, \overline{g})$ has nonpositive Ricci curvature. Then there is no nontrivial pointwise projective transformation from $(M, g)$ to $(M, \overline{g})$.*

**Proof.** Contracting (9) by $g^{ij}$ we get

$$(n-1)\triangle\psi = (n-1)|\nabla\psi|^2 + S - g^{ij}\overline{R_{ij}}. \tag{10}$$

Taking the coordinates $g^{ij}$ diagonalizing $\overline{R_{ij}}$ we see that the invariant $g^{ij}\overline{R_{ij}} \leq 0$. By integrating (10) on $M$, we have $|\nabla\psi| = 0$.

**Corollary 4.** *Let $(M, g)$ and $(M, \overline{g})$ be compact Riemannian manifolds. Assume that $S - g^{ij}\overline{R_{ij}} \geq 0$ or $S - g^{ij}\overline{R_{ij}} \leq 0$ on $(M, g)$. Then there is no nontrivial pointwise projective transformation from $(M, g)$ to $(M, \overline{g})$.*

**Proof.** Take $\psi = -\log f$ for $f > 0$. (10) becomes

$$-(n-1)\triangle f = f(S - g^{ij}\overline{R_{ij}}). \tag{11}$$

Proof follows from the integration of (11) on $M$.

Examining (10), an obstruction to projective transformations between noncompact complete Riemannian manifolds is obtained, which is an extension of Theorem 2.

**Theorem 5.** *Let $(M, g)$ be a noncompact complete Riemannian manifold with scalar curvature $S$, which has at most quadratic volume growth and $\int_M |S^-| dV_g < \infty$. Assume that $\int_M S dV_g$ is infinite or nonnegative, and $(M, \overline{g})$ has nonpositive Ricci curvature. Then there is no nontrivial pointwise projective transformation from $(M, g)$ to $(M, \overline{g})$.*

**Remark.** There are noncompact complete Riemannian manifolds satisfying Theorem 5. Let $(K, h)$ be a compact manifold with nonnegative scalar curvature. Then $K \times R^2$ with the product metric has nonnegative scalar curvature and quadratic volume growth.

*Sketch of proof:* Let us denote $|\Omega| = \int_\Omega 1\ dV_g$ and $X = \nabla\psi$. Then, (10) turns into the following form:

$$(n-1)\ divX = (n-1)|X|^2 + S - g^{ij}\overline{R_{ij}}. \tag{12}$$

Multiplying a compact supported smooth function $\phi^2$ on (12),

$$\int \phi^2 \left((n-1)|X|^2 + S - g^{ij}\overline{R_{ij}}\right) dV_g = -(n-1)\int X(\phi^2) dV_g. \tag{13}$$

Choose $\phi = 1$ on $B(R)$, $\phi = 0$ on the outside of $B(2R)$ and $|\nabla\phi| \le c/R$, where $c$ is a constant and $B(R) \equiv \{x \in M | d(p, x) \le R\}$ for some fixed point $p \in M$.

$$\begin{aligned} 0 &= \int_{B(2R)} \phi^2 \left((n-1)|X|^2 + S - g^{ij}\overline{R_{ij}}\right) + 2(n-1)\phi X \cdot \nabla\phi\ dV_g \\ &\ge \int_{B(2R)} \phi^2((n-1)|X|^2 + S - g^{ij}\overline{R_{ij}})dV_g \\ &\quad -(n-1)\int_{B(2R)-B(R)} |\phi X|^2 + |\nabla\phi|^2 dV_g. \end{aligned}$$

Using quadratic volume growth condition and $\int_M |S^-| dV_g < \infty$, we can show that $|X| \equiv 0$ with some calculations. We refer details to [K].

For manifolds with the volume doubling property, the following obstruction is constructed using the similar method.

**Theorem 6.** *Let $(M, g)$ be a noncompact complete Riemannian manifold which has infinite volume and scalar curvature $S > c_1^2 > 0$ on the outside of any compact subset for a nonzero constant $c_1$, and satisfies that there exists a constant $c_2$ such that $|B(x, 2R)| \le c_2 |B(x, R)|$ for some fixed point $x \in M$ and large $R$. Assume that $(M, \overline{g})$ has nonpositive Ricci curvature $\overline{R_{ij}}$ or $\lim_{R\to\infty} \frac{\int_{B(R)} |(g^{ij}\overline{R_{ij}})^+| dV_g}{Vol B(R)} = 0$. Then there exists no nontrivial pointwise projective transformation from $(M, g)$ to $(M, \overline{g})$.*

## References

[Ei] L.Eisenhart, *Riemannian geometry,* Princeton university, Princeton, 1949.

[Ha] Z. Harél, *Projective mappings and distortion theorems*, J. Diff. Geom. **15** (1980) 97–106.

[K] S. Kim, *Volume and projective mappings between Riemannian manifolds*, preprint.

[Mi] J. Mikeš, *On the existence of $n$-dimensional compact Riemannian spaces that admit non-trivial projective transformations "in the large"*, Dokl. Akad. Nauk SSSR **305** (1989) 534–536.

[Mi1] J. Mikeš, *Geodesic mappings of affine-connected and Riemannian spaces*, J. Math. Sci. New York, (1996), 311-333. Translated from: Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematikai Ee Prilozheniya. Tematicheskie Obzory. Vol. 11, Geometry-2, VINITI, 1994, pp. 112-148.

[MR] J. Mikeš, Z. Radulovi'c, *On projective transformations of Riemannian spaces with harmonic curvature,* in: New developments in differential geometry, Kluwer Acad., Dordrecht, 1999, pp. 279–283.

[Sh] Z. Shen, *On projectively related Einstein metrics in Riemann-Finsler geometry*, Math. Ann. **320** (2001) 625-647.

[Sp] M. Spivak, *A comprehensive introduction to differential geometry* Publish or Perish, Berkeley, 1979.

[Sz] Z. Szabo, *Hilbert's fourth problem*, Adv. Math. **59** (1986) 185-301.

[T] T. Tanno, *On projective transformations of Riemannian manifolds*, in: Geometry and its applications, World Sci., River Edge, 1993, pp. 225–231.

[W] H. Weyl, *Zur Infinitesimalgeometrie: Einordnung der projecktiven und der konformen Auffassung,* Göttingen Nachrichenten, 1921, pp 99-112,

[Y] K. Yamauchi *On infinitesimal projective transformations of a Riemannian manifold with constant scalar curvature*, Hokkaido Math. J. **8** (1979) 167-175.

[Y1] K. Yamauchi, *On Riemannian manifolds admitting infinitesimal projective transformations* Hokkaido Math. J. **16** (1987) 115-125.

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