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ON THE RELATIVE ISOPERIMETRIC INEQUALITY

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Abstract. If $C \subset \mathbf{R}^n$ is a convex domain and D is a subset of $\mathbf{R}^n \sim C$, does D satisfy the isoperimetric inequality $\frac{1}{2}\,n^n\omega_n\text{Volume}(D)^{n-1} \leq \text{Volume}(\partial D \sim \partial C)^n$? Does equality hold if and only if $C = \mathbb{H}$ and D is a half ball with the flat part of its boundary lying in $\partial\mathbb{H}$? This inequality is called the relative isoperimetric inequality. We give three different proofs of the inequality.

1. Introduction

The classical isoperimetric inequality for a domain D in $\mathbf{R}^n$ is

$$n^n\omega_n\text{Volume}(D)^{n-1} \leq \text{Volume}(\partial D)^n. \tag{1}$$

An immediate consequence of this inequality is that if $\mathbb{H}$ is an open half space $\{(x_1, ..., x_n) \in \mathbf{R}^n : x_n > 0\}$ and D is a subset of $\mathbb{H}$ then

$$\frac{1}{2}\,n^n\omega_n\text{Volume}(D)^{n-1} \leq \text{Volume}(\partial D \sim \partial\mathbb{H})^n$$

and equality holds if and only if D is a half ball with the flat part of its boundary contained in $\partial\mathbb{H}$. This follows if one applies (1) to the union of D and its mirror image across $\partial\mathbb{H}$. Then a natural question to ask is the following.

If $C \subset \mathbf{R}^n$ is a convex domain and D is a subset of $\mathbf{R}^n \sim C$, does D satisfy the isoperimetric inequality

$$\frac{1}{2}\,n^n\omega_n\text{Volume}(D)^{n-1} \leq \text{Volume}(\partial D \sim \partial C)^n? \tag{2}$$

Does equality hold if and only if $C = \mathbb{H}$ and D is a half ball with the flat part of its boundary lying in $\partial\mathbb{H}$?

(2) is called the *relative isoperimetric inequality*, C is called the supporting set of D, and $\text{Volume}(\partial D \sim \partial C)$ is called the relative boundary volume of D. For $n = 2$ it is easy to prove (2): just reflect the convex hull of D about its linear boundary and apply (1).

The first partial answer for $n \geq 3$ was obtained by I. Kim [K2]. He showed that if $U = \{(x, y) \in \mathbf{R}^2 : y \geq f(x), f'' \geq 0\}$, then (2) holds for $C = U \times \mathbf{R}^{n-2}$. Recently the author showed that the relative isoperimetric inequality holds if ∂C is a graph over a horizontal hyperplane and C, D are symmetric about $n-1$ mutually orthogonal vertical hyperplanes of $\mathbf{R}^n$ [C5]. In particular, (2) holds when C is a ball. Finally the complete proof of the relative isoperimetric inequality has been obtained

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by the author and Ritoré [CR]. In this paper we shall summarize the proofs of these three results. These proofs use a variety of interesting methods of geometry; the Alexandrov space, Gromov's volume preserving map, Steiner's symmetrization, and Almgren's isoperimetric profile approach. In the last section we shall present the relative isoperimetric inequality on a nonpositively curved surface.

2. Alexandrov Space

When we prove the relative isoperimetric inequality for a domain D in the half space $\mathbb{H}$, we take the union $\widetilde{D}$ of D and its mirror image across $\partial\mathbb{H}$; take $\mathbf{R}^n$ as the union of $\mathbb{H}$ and its mirror image; then the classical isoperimetric inequality for $\widetilde{D}$ gives the desired relative isoperimetric inequality for D. I. Kim's idea in [K2] is to imitate this argument on the domain D outside an open convex subset C in $\mathbf{R}^n$. To do so, it is necessary to introduce $\widetilde{D}$ as the double cover of D and $\widetilde{C}$ as the double cover of $\mathbf{R}^n \sim C$. More precisely, we define $\widetilde{D}$ to be the disjoint union of D and itself with the identification along $\partial D \cap \partial C$ and define $\widetilde{C}$ to be the disjoint union of $\mathbf{R}^n \sim C$ and itself with the identification along ∂C.

The first observation of I. Kim is that both $\mathbf{R}^n \sim C$ and $\widetilde{C}$ are nonpositively curved in the sense of Alexandrov as follows. Let X be a length space, i.e., a locally compact complete metric space such that given two points p and q in X, there exists a shortest curve (geodesic) between them. X is said to have nonpositive curvature if for any $p \in X$ there exists a neighborhood U of p such that for any geodesic triangle $\triangle pqr \subset U$, the geodesic triangle $\triangle\bar{p}\bar{q}\bar{r} \subset \mathbf{R}^2$ of the same edge lengths as $\triangle pqr$ satisfies the following: For any $s \in \overline{qr}$ and $\bar{s} \in \overline{\bar{q}\bar{r}}$ with $d(q,s) = d(\bar{q},\bar{s})$ we have

$$d(p,s) \leq d(\bar{p},\bar{s}). \tag{3}$$

Here let us assume that $C = U \times \mathbf{R}^{n-2}$, $U = \{(x,y) \in \mathbf{R}^2 : y > f(x), f'' \geq 0\}$. Let $\triangle$ be a triangle in $\mathbf{R}^n \sim C$. If at most one vertex of $\triangle$ lies in ∂C then $\triangle$ is Euclidean and so equality holds in (3). Suppose that all three vertices of $\triangle$ lie in ∂C. Then $\triangle$ lies entirely in ∂C. Moreover ∂C is isometric to $\mathbf{R}^{n-1}$ and hence $\triangle$ is Euclidean; so (3) holds again. Suppose now two vertices of $\triangle$ lie in ∂C. Then the edge between the two vertices are concave in $\triangle$. Hence (3) clearly holds. Therefore $\mathbf{R}^n \sim C$ is nonpositively curved. Thus by Corollary 5 of [Ba] $\widetilde{C}$ is also nonpositively curved.

$\widetilde{D}$ can be thought of as a subset of $\widetilde{C}$. We claim that $\widetilde{D}$ satisfies the isoperimetric inequality (1). For this claim we need a theorem by Cao and Escobar.

Theorem 1. [CE] *Let M^n be a complete simply connected nonpositively curved piecewise linear manifold. Then for any domain D with rectifiable boundary ∂D, the classical isoperimetric inequality*

$$n^n\omega_n \mathrm{Volume}(D)^{n-1} \leq \mathrm{Volume}(\partial D)^n$$

holds with equality if and only if D is isometric to a Euclidean ball.

Since $\widetilde{C}$ can be arbitrarily approximated by a piecewise linear manifold Theorem 1 implies

$$n^n\omega_n \mathrm{Volume}(\widetilde{D})^{n-1} \leq \mathrm{Volume}(\partial\widetilde{D})^n.$$

Note that

$$\text{Volume}(\widetilde{D}) = 2\,\text{Volume}(D), \quad \text{Volume}(\partial\widetilde{D}) = 2\,\text{Volume}(\partial D \sim \partial C).$$

Thus we get the following.

Theorem 2. [K2] *Assume that $C = U \times \mathbf{R}^{n-2}$, $U = \{(x,y) \in \mathbf{R}^2 : y > f(x)$ for a convex function $f\}$. Then a domain $D \subset \mathbf{R}^n$ outside C satisfies*

$$\frac{1}{2}\, n^n \omega_n \text{Volume}(D)^{n-1} \leq \text{Volume}(\partial D \sim \partial C)^n,$$

where equality holds if and only if D is a half ball and $\partial D \cap \partial C$ is a subset of a hyperplane.

3. Gromov's Method

In [Gr] Gromov gave a new proof of the classical isoperimetric inequality. As F. Morgan pointed out to us, Knothe [Kn] and Berger [Bg] also used the same method as Gromov. His proof is based on a volume-preserving map whose divergence is bigger than or equal to the dimension of space. Here we shall see how Gromov's method can be adapted for our purpose and why the convexity of the supporting set is necessary. Just in a heuristic way, let us briefly show how to construct a volume-preserving map ψ from $D \subset \mathbf{R}^2$ onto a half disk $B = \{(x,y) \in \mathbf{R}^2 : x^2+y^2 < r^2, y > 0\}$ with $\text{Area}(D) = \text{Area}(B)$. Suppose $\psi(a,b) = (\psi_1, \psi_2)$. Then ψ_1 and ψ_2 are chosen in such a way that $\text{Area}\{(x,y) \in D : x \leq a\} = \text{Area}\{(x,y) \in B : x \leq \psi_1\}$ and $\text{Length}\{(x,y) \in D : x = a, y \leq b\}\, da = \text{Length}\{(x,y) \in B : x = \psi_1, y \leq \psi_2\}\, d\psi_1$. Consequently ψ is area-preserving (i.e., $\det D\psi = 1$) and $D\psi$ is lower triangular.

Theorem 3. [C5] *Let C be a convex domain in $\mathbf{R}^n$ and D a subset of $\mathbf{R}^n \sim C$ with piecewise C^1 boundary. Suppose that every normal vector η to $\partial D \cap \partial C$ toward the exterior of D does not point upward, that is, $\langle \eta, \frac{\partial}{\partial x^n} \rangle \leq 0$ for the unit vertical vector $\frac{\partial}{\partial x^n}$. Suppose also that there exist vertical hyperplanes $\Pi_1, ..., \Pi_{n-1}$ which are mutually perpendicular such that C and D are symmetric about each of them. Then*

$$\frac{1}{2}\, n^n \omega_n \text{Volume}(D)^{n-1} \leq \text{Volume}(\partial D \sim \partial C)^n,$$

where equality holds if and only if D is a half ball.

Partial Proof. First let us define a C^1 map $\phi_D : D \to [0,1]^n$ by

$$\phi_D(x_1, ..., x_n) = (\phi_1, ..., \phi_n), \ \phi_i = \frac{\overline{v}_i}{v_i},$$

$$\begin{aligned} v_i = L^{n-i+1}\{(a_1, ..., a_n) \in D \ &: \ a_j = x_j, 1 \leq j \leq i-1, -\infty \leq a_k \leq \infty, i \leq k \leq n\}, \\ \overline{v}_i = L^{n-i+1}\{(a_1, ..., a_n) \in D \ &: \ a_j = x_j, 1 \leq j \leq i-1, -\infty \leq a_i \leq x_i, \\ &\qquad -\infty \leq a_k \leq \infty, i+1 \leq k\}, \end{aligned}$$

where L^k is the k-dimensional Lebesgue measure. Then $\phi_i = \phi_i(x_1, ..., x_i)$ and the Jacobian matrix of ϕ_D, $\left(\frac{\partial \phi_i}{\partial x_j}\right)$, is lower triangular with diagonal entries $\frac{\partial \phi_i}{\partial x_i} = \frac{v_{i+1}}{v_i}$ and $\frac{\partial \phi_n}{\partial x_n} = \frac{1}{v_n}$. Therefore

$$\det\left(\frac{\partial \phi_i}{\partial x_j}\right) = \frac{1}{v_1}.$$

Similarly, define $\phi_B : B \to [0,1]^n$ where B is the half ball

$$\{(x_1, ..., x_n) \in \mathbf{R}^n : x_n \geq 0, \sum x_i^2 \leq (2\omega_n^{-1}\text{Volume}(D))^{2/n}\}. \tag{4}$$

Note that Volume(B) = Volume$(D) = v_1$. Like ϕ_D the Jacobian determinant of ϕ_B equals $1/v_1$. Let $\psi : D \to B$ be defined by $\psi = \phi_B^{-1} \circ \phi_D$. Then the Jacobian determinant of ψ equals 1. In other words, ψ is a volume-preserving map.

Now let us consider a vector field V on D defined by $V(x) =$ *the position vector of* $\psi(x), x \in D$. Since the Jacobian matrix of ψ is also lower triangular, it follows from the arithmetic-geometric mean inequality that

$$n = n(\det D\psi)^{1/n} \leq \text{div} V. \tag{5}$$

Let Π_n be the horizontal hyperplane $\{x_n = 0\}$ and let $U_1, ..., U_{2^{n-1}}$ be the congruent subsets of Π_n separated by the vertical hyperplanes $\Pi_1, ..., \Pi_{n-1}$. Translating C and D in a suitable way we may assume that each Π_i contains $(0, ..., 0)$. Define the projection $p : \mathbf{R}^n \to \Pi_n$ by $p(x_1, ..., x_n) = (x_1, ..., x_{n-1}, 0)$. By the divergence theorem applied to (5)

$$n\text{Volume}(D) \leq \int_{\partial D \sim \partial C} \langle V, \eta \rangle + \int_{\partial D \cap \partial C} \langle V, \eta \rangle, \tag{6}$$

where η is the outward unit normal to ∂D. By (4) we have

$$|V| \leq (2\,\omega_n^{-1}\text{Volume}(D))^{1/n} \text{ on } \partial D \sim \partial C. \tag{7}$$

By the symmetry of C and D about $\Pi_1, ..., \Pi_{n-1}$ and by the convexity of C we get

$$\langle V, \eta \rangle \leq 0 \text{ on } \partial D \cap \partial C. \tag{8}$$

This is because if $x \in \partial D \cap \partial C$ and $p(x) \in U_k, 1 \leq k \leq 2^{n-1}$, then both $\psi(x)$ and $-p(q_\eta)$ lie in U_k, where $q_\eta \in \mathbf{R}^n$ is the point whose position vector is η. Therefore it follows from (6), (7), and (8) that

$$n\text{Volume}(D) \leq (2\,\omega_n^{-1}\text{Volume}(D))^{1/n}\text{Volume}(\partial D \sim \partial C),$$

which imples (2).

See [C5] for the case of equality. □

4. Steiner's Symmetrization

One of the oldest and most powerful methods in proving isoperimetric inequalities is Steiner's symmetrization [S2]. The key idea of this method is that given k functions $x_n = f_1(x_1, ..., x_{n-1})$, ..., $x_n = f_k(x_1, ..., x_{n-1})$, the volume of the graph of the average function of $f_1, ..., f_k$ is not bigger than the average of the volumes of the graphs of $f_1, ..., f_k$. This volume estimate is based on the simple inequality for k vectors in $\mathbf{R}^n$: $|v_1 + ... + v_k| \leq |v_1| + ... + |v_k|$. Here, using the symmetrization method, we shall improve Theorem 3.

Theorem 4. [C5] *Let C be a convex domain in $\mathbf{R}^n$, D a subset of $\mathbf{R}^n \sim C$ with piecewise C^1 boundary, and Π_n a horizontal hyperplane $\{x_n = 0\}$. Suppose that both $\partial D \sim \partial C$ and $\partial D \cap \partial C$ are graphs over a closed set $A \subset \Pi_n$. If A is symmetric about $n-1$ vertical hyperplanes $\Pi_1, ..., \Pi_{n-1}$ which are mutually perpendicular, then*

$$\frac{1}{2}\, n^n \omega_n \mathrm{Volume}(D)^{n-1} \leq \mathrm{Volume}(\partial D \sim \partial C)^n,$$

where equality holds if and only if D is a half ball.

Although the symmetry assumption is required in Theorems 3 and 4, it is not necessary in case the convex set C is a ball:

Theorem 5. [C5] *If C is a ball in $\mathbf{R}^n$ and D is a subset of $\mathbf{R}^n \sim C$ with rectifiable boundary, then*

$$\frac{1}{2}\, n^n \omega_n \mathrm{Volume}(D)^{n-1} \leq \mathrm{Volume}(\partial D \sim \partial C)^n$$

with equality if and only if D is a half ball.

It is easy to prove this theorem once we know that the isoperimetric region of the complement of a ball is rotationally symmetric about a line through the center of the ball.

Lemma 1. [C5] *Outside a ball $C \subset \mathbf{R}^n$ there exists a set $\widetilde{D}$ whose boundary has the least relative volume* $\mathrm{Volume}(\partial\widetilde{D} \sim \partial C)$ *among all sets outside C with the same volume as $\widetilde{D}$. In fact, $\partial\widetilde{D} \sim \partial C$ is a spherical cap perpendicular to ∂C and $\partial\widetilde{D} \cap \partial C$ lies in an open hemisphere of ∂C.*

5. Isoperimetric Profile

The author and M. Ritoré [CR] have recently proved (2) in the general case. Here we outline the idea of the proof. Given a convex domain $C \subset \mathbf{R}^n$, define the *relative isoperimetric profile* of $\mathbf{R}^n \sim \overline{C}$, $I_C : \mathbb{R}^+ \to \mathbb{R}^+$, by

$$I_C(V) = \inf_D \{\mathrm{Area}(\partial D \sim \partial C) : D \subset \mathbf{R}^n \sim \overline{C},\ \mathrm{Volume}(D) = V\}.$$

For the upper half space $\mathbb{H}$ the relative isoperimetric profile of $\mathbf{R}^n \sim \mathbb{H}$ can be explicitly computed by

$$I_{\mathbb{H}}(V) = n\left(\frac{\omega_n}{2}\right)^{\frac{1}{n}} V^{\frac{n-1}{n}},$$

and the relative isoperimetric inequality (2) is equivalent to

$$I_{\mathbb{H}}(\mathrm{Volume}(D)) \leq I_C(\mathrm{Volume}(D)), \tag{9}$$

with quality if and only if D is a half ball. How can one prove (9)? The key idea comes from the Gauss map $\nu : \partial D \sim \partial C \to S^{n-1}$. For the proof of (9) one needs to show that the Gauss map covers more than half of S^{n-1} in case $\partial D \sim \partial C$ is perpendicular to ∂C. More precisely, one needs

$$\frac{n\omega_n}{2} \leq \int_{\partial D \sim \partial C} GK,$$

where GK is the Gauss-Kronecker curvature (the product of principal curvatures). Then using the arithmetic-geometric mean inequality between the mean curvature H (the arithmetic mean of principal curvatures) and GK

$$GK \leq H^{n-1},$$

one gets

$$\frac{n\omega_n}{2} \leq \int_{\partial D \sim \partial C} GK \leq \int_{\partial D \sim \partial C} H^{n-1} \leq (\sup_{\partial D \sim \partial C} H)^{n-1} \,\mathrm{Volume}(\partial D \sim \partial C).$$

Hence

$$H_0(\mathrm{Volume}(\partial D \sim \partial C)) \leq \sup_{\partial D \sim \partial C} H, \tag{10}$$

where $H_0(\mathrm{Volume}(\partial D \sim \partial C))$ is the mean curvature of the hemisphere of volume $\mathrm{Volume}(\partial D \sim \partial C)$. What does (10) mean? To understand it one first needs to note that the first variation formula for volume implies

$$\frac{dI}{dv} = (n-1)H.$$

Secondly, foliate the first quadrant of $\mathbf{R}^2$ by the graph of $I_{\mathbb{H}}(V)$ and all its translates along the V-axis. Then (10) implies that the graph of $I_C(V)$ crosses the translates of the graph of $I_{\mathbb{H}}(V)$ monotonically. Therefore (9) follows and one has the following.

Theorem 6. *If D is a domain outside a convex domain C in $\mathbf{R}^n$ then D satisfies*

$$\frac{1}{2}\, n^n \omega_n \mathrm{Volume}(D)^{n-1} \leq \mathrm{Volume}(\partial D \sim \partial C)^n,$$

with equality if and only if D is a half ball with the flat part of its boundary lying in $\partial\mathbb{H} \cap \partial C$.

It would be interesting if one could derive a version of the relative isoperimetric inequality for minimal surfaces. Therefore one can propose the following.

Open Problem 1. *Given a convex domain C in $\mathbf{R}^n$ and an m-dimensional minimal submanifold N outside C such that N is orthogonal to ∂C along $\partial N \cap \partial C$, prove that*

$$\frac{1}{2}\, m^m \omega_m \mathrm{Volume}(N)^{m-1} \leq \mathrm{Volume}(\partial N \sim \partial C)^m,$$

where equality holds if and only if N is a half ball.

I. Kim [K1] obtained a partial result for this open problem when N is two-dimensional. He proved that if S is a minimal surface in a Riemannian manifold M of constant sectional curvature $K \leq 0$, S lies outside a convex domain C in M and is orthogonal to ∂C, and $\partial S \sim \partial C$ is connected or radially connected from a point p of $\partial S \cap \partial C$ (i.e., the set $\{d(p,q) : q \in \partial S \sim \partial C\}$ is connected), then

$$2\pi \,\mathrm{Area}(S) \leq \mathrm{Length}(\partial S \sim \partial C)^2 + K\,\mathrm{Area}(S)^2$$

and equality holds if and only if S is a totally geodesic half disk. For the proof of this, he first showed that

$$\mathrm{Area}(S) \leq \mathrm{Area}(p ⋇ (\partial S \sim \partial C)) \text{ for any } p \in C$$

and

$$\mathrm{Angle}(\partial S \sim \partial C, p) \geq \pi \text{ for any } p \in \partial S \cap \partial C.$$

Then he used the method of developing, cutting and pasting as in [C1].

6. Negatively Curved Surfaces

It was Carleman [Ca] who first showed that the classical isoperimetric inequality $4\pi A \leq L^2$ still holds for some curved surfaces: disk type minimal surfaces in space. Then in 1926 Weil [We] obtained the same result for disk type surfaces of negative Gaussian curvature. Thereafter a variety of different methods were employed by a dozen mathematicians to prove the same or more general inequalities; Bol [Bo] used parallel curves and Alexandrov [Ax] used the method of polyhedral approximation. Huber [Hu] improved the inequality of Carleman and its generalization to subharmonic functions by Beckenbach and Radó [BR].

In addition to these various proofs we can obtain a new proof if we apply the maximum principle to the Dirichlet boundary value problem for the Laplacian [C6]. By applying the mixed boundary value problem, instead, we can get a relative isoperimetric inequality as follows.

Theorem 7. [C5] *Let S be a disk type surface of nonpositive Gaussian curvature. Suppose that ∂S is the disjoint union of Γ_1 and Γ_2 such that Γ_1 is connected and concave, i.e., if $c(s)$ is an arclength parametrization of Γ_1, then $c''(s)$ vanishes or points outward from S. Then*

$$2\pi \mathrm{Area}(S) \leq \mathrm{Length}(\Gamma_2)^2 \tag{11}$$

and equality holds if and only if S is a flat half disk.

Proof. Let x and y be isothermal coordinates on S. Then the metric and the Gaussian curvature K of S can be written as

$$ds^2 = e^{2\lambda}(dx^2 + dy^2), \ K = -e^{-2\lambda} \triangle \lambda. \tag{12}$$

By the curvature assumption we have

$$\triangle \lambda \geq 0 \text{ on } S.$$

The concavity of the free part Γ_1 implies

$$\frac{\partial \lambda}{\partial \nu} \leq 0,$$

where ν is the outward unit normal to Γ_1. This is because

$$\begin{aligned} 0 &\geq \left\langle \nabla_{e^{-\lambda}\frac{\partial}{\partial y}} e^{-\lambda}\frac{\partial}{\partial y}, e^{-\lambda}\frac{\partial}{\partial x} \right\rangle = -\left\langle e^{-\lambda}\frac{\partial}{\partial y}, \nabla_{e^{-\lambda}\frac{\partial}{\partial y}} e^{-\lambda}\frac{\partial}{\partial x} \right\rangle \\ &= -e^{-3\lambda}\left\langle \frac{\partial}{\partial y}, \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} \right\rangle = -e^{-3\lambda}\left\langle \frac{\partial}{\partial y}, \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} \right\rangle \\ &= -\frac{1}{2}e^{-3\lambda}\frac{\partial}{\partial x}\left|\frac{\partial}{\partial y}\right|^2 = -e^{-\lambda}\frac{\partial \lambda}{\partial x} = \frac{\partial \lambda}{\partial \nu}. \end{aligned}$$

Solve the mixed boundary value problem

$$\triangle h = 0 \text{ on } S, \ h = \lambda \text{ on } \Gamma_2 \text{ and } \frac{\partial h}{\partial \nu} = 0 \text{ on } \Gamma_1.$$

Let us now introduce a surface $\widetilde{S}$ which is S equipped with the flat metric $\widetilde{g} = e^{2h}(dx^2 + dy^2)$. Actually $\widetilde{S}$ is the image of S in the complex plane under the holomorhic map $\phi(z)$ such that $\log|\phi'(z)| = h(x,y), z = x+iy$. Note that the boundary condition on Γ_2 implies Length($\widetilde{\Gamma}_2$) = Length(Γ_2) and the condition on Γ_1 implies that $\widetilde{\Gamma}_1$ is a line segment in $\widetilde{S}$. From the maximum principle we get $h \geq \lambda$ and so Area($\widetilde{S}$) $\geq$ Area(S). Therefore (11) follows from the relative isoperimetric inequality for $\widetilde{S} \subset \mathbf{R}^2$. See [C5] for more details. □

In this section we have seen that nonpositively curved two-dimensional surfaces satisfy the same isoperimetric inequality as in $\mathbf{R}^2$. In regard to higher dimensional nonpositively curved Riemannian manifolds, Aubin conjectured that in the sense of the isoperimetric inequality, $\mathbf{R}^n$ is more efficient than any complete simply connected Riemannian manifold M^n of nonpositive sectional curvature. More precisely, he conjectured that for any domain D in M^n

$$n^n\omega_n \text{Volume}(D)^{n-1} \leq \text{Volume}(\partial D)^n$$

and equality holds if and only if D is a Euclidean ball. Recently Kleiner [Kl] and Croke [Cr] proved this inequality in M^3 and M^4, respectively; but this conjecture is still open for $n \geq 5$. Extending Aubin's conjecture to the case of relative isoperimetric inequality, we would like to propose the following:

Open Problem 2. *Let C be a convex domain in a complete simply connected Riemannian manifold M^n of nonpositive sectional curvature and D a subset of $M \sim C$. Prove that*

$$\frac{1}{2}\, n^n\omega_n \text{Volume}(D)^{n-1} \leq \text{Volume}(\partial D \sim \partial C)^n,$$

where equality holds if and only if D is a Euclidean half ball.

We should mention that this open problem has been solved for $n = 3$ in [CR].

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