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VARIATIONAL PROPERTIES OF HARMONIC RIEMANNIAN FOLIATIONS

KYOUNG HEE HAN AND HOBUM KIM

Abstract. We obtain a second variation formula for the energy functional for a harmonic Riemannian foliation in terms of the second and third fundamental tensors, integrability tensor and curvature regarding the foliation. From this formula, we conclude that if the second fundamental tensor and the integrability tensor of a harmonic Riemannian foliation is small compared with the partial Ricci tensor, then it is stable.

1. Introduction.

A foliation $\mathcal{F}$ on a Riemannian manifold (M, g_M) is called Riemannian and g_M bundle-like if all the leaves are locally equi-distant to each other. Such a foliation is characterized by the property that a geodesic orthogonal to the foliation at one point is orthogonal everywhere. $\mathcal{F}$ is said to be harmonic if all the leaves of $\mathcal{F}$ are minimal submanifolds([4]).

F. Kamber and Ph. Tondeur([4],[5],[6]) showed that a Riemannian foliation is harmonic if and only if it is critical for the energy functional under an appropriate class of so-called special variations defined by sections of the normal bundle([4]). They also derived a second variational formula for the energy functional and considered the stability of a harmonic Riemannian foliations ([5],[6]).

In this note, exploiting the second variation formula by Kamber and Tondeur([5]), we obtain a second variation formula for the energy functional for a harmonic Riemannian foliation in terms of the second fundamental tensor, integrability tensor and curvature regarding the foliation. As an immediate consequence it will follow that if the second fundamental tensor and the integrability tensor of a harmonic Riemannian foliation is small compaired with the partial Ricci tensor, then it is stable. This result manifests the fact that the second fundamental tensor, integrability tensor and curvature reflects fundamental aspects of geometry of foliations.

2. Energy of a foliation

Let $\mathcal{F}$ be a foliation of dimension p and codimension q on a smooth manifold of dimension n.

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The *tangent bundle* L of a foliation $\mathcal{F}$ is the subbundle of TM, consisting of all vectors tangent to the leaves of $\mathcal{F}$. The *normal bundle* Q of codimension-q foliation $\mathcal{F}$ on M is the quotient bundle $Q = TM/L$.

The tangent bundle L of $\mathcal{F}$ is integrable or involutive and Q appears in exact sequence of vector bundles

$$0 \longrightarrow L \longrightarrow TM \xrightarrow{\pi} Q \longrightarrow 0. \tag{2.1}$$

Consider a Riemannian metric g_M on M. Then TM splits orthogonally as $TM = L \oplus L^{\perp}$ with $\sigma : Q \to L^{\perp} \subset TM$ splitting the sequence (2.1). The metric g_M on TM is then a direct sum $g_M = g_L \oplus g_{L^{\perp}}$. With $g_Q = \sigma^* g_{L^{\perp}}$, the splitting map $\sigma : (Q, g_Q) \to (L^{\perp}, g_{L^{\perp}})$ is a metric isomorphism.

Let $\overset{\circ}{\nabla}$ denote the *(partial) Bott connection* in Q defined by

$$\overset{\circ}{\nabla}_X s = \pi[X, Y_s] \tag{2.2}$$

for $X \in \Gamma L, Y_s \in \Gamma TM$ with $\pi(Y_s) = s$. We observe that the RHS in (2.2) is independent of the choice of Y_s.

Let ∇^M denote the Levi-Civita connection associated to the Riemannian metric g_M on M. A connection ∇ in Q is defined by

$$\nabla_X s = \begin{cases} \pi[X, Y_s] & \text{for } X \in \Gamma L \\ \pi(\nabla^M_X Y_s) & \text{for } X \in \Gamma \sigma Q, \end{cases} \tag{2.3}$$

where $s \in \Gamma Q$ and $Y_s = \sigma(s) \in \Gamma \sigma Q$.

The first condition says that ∇ is an adapted connection on Q, i.e. a connection extending the (partial) Bott connection $\overset{\circ}{\nabla}$ along L.

A foliation $\mathcal{F}$ is *Riemannian* (or an *R-foliation*), if the normal bundle is equipped with a holonomy invariant fiber metric g_Q. This condition is expressed in terms of the Bott connection $\overset{\circ}{\nabla}$ by $\overset{\circ}{\nabla}_X g_Q = 0$ for $X \in \Gamma L$.

A Riemannian metric g_M on M is *bundle-like* with respect to the foliation $\mathcal{F}$, if the fiber metric g_Q induced on Q turns the foliation into an R-foliation. An R-foliation admits a bundle-like metric.

Let $\mathcal{F}$ be an R-foliation with a metric g_Q on the normal bundle Q. For $r \geq 0$, We denote by $\Omega^r(M, Q)$ the space of all smooth Q-valued r-forms on M. For a connection ∇ on Q, we consider the exterior derivative $d_\nabla : \Omega^r(M, Q) \to \Omega^{r+1}(M, Q), \quad r \geq 0$, given by for $\omega \in \Omega^r(M, Q)$,

$$\begin{aligned} (d_\nabla \omega)(X_1, \cdots, X_{r+1}) &= \sum_{i=1}^{r+1} (-1)^{i+1} \nabla_{X_i} \omega(X_1, \cdots, \hat{X}_i, \cdots, X_{r+1}) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{r+1}), \end{aligned} \tag{2.4}$$

where the X_i's are vector fields on M.

The *covariant derivative* $\nabla_X : \Omega^r(M,Q) \to \Omega^r(M,Q), \quad X \in \Gamma TM, \quad r \geq 0$, is given by

$$(2.5)\qquad \begin{aligned}(\nabla_X \omega)(X_1, \cdots, X_r) =& \nabla_X \omega(X_1, \cdots, X_r) \\ &- \sum_{i=1}^{r} \omega(X_1, \cdots, \nabla^M_X X_i, \cdots, X_r),\end{aligned}$$

where $\omega \in \Omega^r(M,Q)$ and $X_i \in \Gamma TM$.

Since ∇^M is torsion-free, $d_\nabla \omega$ can be expressed as

(2.6)

$$(d_\nabla \omega)(X_1, \cdots, X_{r+1}) = \sum_{i=1}^{r+1} (-1)^{i+1} (\nabla_{X_i} \omega)(X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_{r+1}).$$

For the case $r = 1$, this formula reads

$$(2.7)\qquad (d_\nabla \omega)(X,Y) = (\nabla_X \omega)(Y) - (\nabla_Y \omega)(X).$$

The Q-valued bilinear form on M

$$\alpha(X,Y) = -(\nabla \pi)(X,Y) = \pi(\nabla^M_X Y) - \nabla_X \pi(Y)$$

for $X, Y \in \Gamma TM$ is called *the second fundamental form of the foliation.*

For each $\nu \in \Gamma Q$ the map $W(\nu) : L \to L$ defined by

$$g_M(W(\nu)U, V) = g_Q(\alpha(U,V), \nu)$$

for $U, V \in \Gamma L$ is called the *Weingarten operator*. $\mathcal{F}$ is said to be *harmonic* if $\mathrm{Trace}\, \mathrm{W}(\nu) = 0$ for each $\nu \in \Gamma Q$.

The harmonicity of the foliation is related with the Laplacian

$$(2.8)\qquad \Delta = d_\nabla d_\nabla^* + d_\nabla^* d_\nabla$$

where the star operator on M extends to Q-valued forms

$$* : \Omega^r(M,Q) \to \Omega^{n-r}(M,Q)$$

and the codifferential $d_\nabla^* : \Omega^r(M,Q) \to \Omega^{r-1}(M,Q), r > 0$ of the exterior differential d_∇ is given in terms of the star operator by

$$(2.9)\qquad d_\nabla^* \omega = (-1)^{n(r+1)+1} * d_\nabla * \omega, \quad \omega \in \Omega^r(M,Q).$$

The codifferential d_∇^* becomes the formal adjoint of d_∇ with respect to the naturally induced scalar product

$$\langle \mu, \nu \rangle = \int_M g_Q(\mu, \nu) \cdot \eta_M \tag{2.10}$$

of sections $\mu, \nu \in \Gamma Q$ on Q-valued forms over a compact oriented manifold M. Here η_M is the volume form associated to the metric g_M on M. The kernel of the Laplacian Δ coincides precisely with the forms which are both d_∇-closed and d_∇^*-closed. A foliation is harmonic if and only if $\Delta\pi = 0$([4]).

The *energy* of the foliation $\mathcal{F}$ is

$$E(\mathcal{F}) = \frac{1}{2} \parallel \pi \parallel^2 \tag{2.11}$$

where $\pi \in \Omega^1(M, Q)$ is the canonical projection $\pi : TM \to Q$ and the norm $\| \cdot \|$ is given by the scalar product $\langle\ ,\ \rangle$ on $\Omega^r(M, Q)$, $\quad r \geq 1$ such that

$$\langle \omega, \omega' \rangle = \int_M g_Q(\omega \wedge *\omega'). \tag{2.12}$$

3. Second variation formula and stability of a harmonic Riemannian foliation.

Assume g_M to be bundle-like. A section $\nu \in \Gamma Q$ defines a *special variation* $\mathcal{F}_t$ of $\mathcal{F}_0 = \mathcal{F}$ through Riemannian foliations by patching the local data

$$\Phi_t^\alpha(x) = \exp_{f^\alpha(x)}(t\nu^\alpha(x)), \tag{3.1}$$

where $f^\alpha$is a local submersion defining $\mathcal{F}$ in an open set U^α. Φ_t^α is then the local submersion defining $\mathcal{F}_t$ for $| t | \leq \epsilon$, where $\epsilon > 0$ is sufficiently small. The RHS in (3.1) denotes the endpoint of the geodesic segment starting at $f^\alpha(x)$ and determined by $t\nu^\alpha(x)$. This is the construction of Eells-Sampson. Clearly

$$\nu^\alpha(x) = \left.\frac{d}{dt}\right|_{t=0} \Phi_t^\alpha(x). \tag{3.2}$$

The following facts are due to Kamber and Tondeur([4],[5]).

Lemma 3.1. *Let M be a compact oriented manifold with a Riemannian foliation $\mathcal{F}$ and bundle-like metric g_M. Then $\mathcal{F}$ is harmonic if and only if it is an extremal of the energy functional for special variations of $\mathcal{F}$.*

Lemma 3.2(Second variation formula). *Let M be a compact oriented manifold and $\mathcal{F}$ a harmonic Riemannian foliation with respect to a bundle-like metric g_M. Consider the 2-parameter family $\mathcal{F}_{s,t}$ of special variations of $\mathcal{F} = \mathcal{F}_{0,0}$ defined by two sections μ, ν of the normal bundle Q. Then for the second derivative of the energy we have*

$$(\partial^2/\partial s \partial t) E(\mathcal{F}_{s,t})\big|_{s=0,t=0} = \langle (\Delta - \rho_\nabla)\mu, \nu \rangle, \tag{3.3}$$

where ρ_∇ is the Ricci operator given by $(\rho_\nabla \mu)_x = \sum_{\gamma=p+1}^{n} R_\nabla(\mu, e_\gamma) e_\gamma$ for an orthonormal basis $\{e_{p+1}, \cdots, e_n\}$ of Q_x.

Theorem 3.3. *Let $\mathcal{F}$ be a harmonic Riemannian foliation of dimension p with normal bundle Q on a compact oriented Riemannian manifold M of dimension n. Let ∇^M be the Levi-Civita connection on M and ∇ the connection on Q defined by (2.3). Consider the family $\mathcal{F}_t$ of special variations of $\mathcal{F} = \mathcal{F}_0$ defined by $\nu \in \Gamma Q$. Then for the second derivative of the energy we have*

$$\begin{aligned} &\frac{d^2}{dt^2}\bigg|_{t=0} E(\mathcal{F}_t) \\ &\qquad = \int_M g_M(\Delta_M \nu, \nu) - \| W(\nu) \|^2 - \| A(\nu) \|^2 - g_Q(\rho_\nabla \nu, \nu), \end{aligned} \tag{3.4}$$

where W is the Weingarten operator and A is the integrability tensor given by

$$A_E F = \pi^\perp(\nabla^M_{\pi(E)} \pi(F)) + \pi(\nabla^M_{\pi(E)} \pi^\perp(F)) \tag{3.6}$$

for $E, F \in \Gamma TM$.

Proof. Consider the family $\mathcal{F}_t$ of special variations of $\mathcal{F} = \mathcal{F}_0$ defined by the section $\nu \in \Gamma Q$. Let g_M be a bundle-like metric and $E_1, \cdots, E_n$ an orthonormal local frame of TM on a neighborhood of $x \in M$ such that $E_1, \cdots, E_p \in \Gamma L$ and $E_{p+1}, \cdots, E_n \in \Gamma Q$. From (3.3), we have

$$\begin{aligned} \frac{d^2}{dt^2}\bigg|_{t=0} E(\mathcal{F}_t) &= \langle (\Delta - \rho_\nabla)\nu, \nu \rangle \\ &= \int_M \sum_{A=1}^{n} g_Q(\nabla_{E_A} \nu, \nabla_{E_A} \nu) - g_Q(\rho_\nabla \nu, \nu). \end{aligned} \tag{3.7}$$

Using the integrability tensor A and the Weingarten operator W, we find that

$$\begin{aligned}
&\sum_{A=1}^{n} g_Q(\nabla_{E_A}\nu, \nabla_{E_A}\nu) \\
&= \sum_{i=1}^{p} g_Q(\nabla_{E_i}\nu, \nabla_{E_i}\nu) + \sum_{\alpha=p+1}^{n} g_Q(\nabla_{E_\alpha}\nu, \nabla_{E_\alpha}\nu) \\
&= \sum_{i=1}^{p} g_M(\nabla^M_{E_i}\nu, \nabla^M_{E_i}\nu) - \sum_{i=1}^{p} g_L(\pi^\perp(\nabla^M_{E_i}\nu), \pi^\perp(\nabla^M_{E_i}\nu)) \\
&\quad - \sum_{i=1}^{p} g_Q(\pi(\nabla^M_{\nu}E_i), \pi(\nabla^M_{\nu}E_i)) + \sum_{\alpha=p+1}^{n} g_M(\nabla^M_{E_\alpha}\nu, \nabla^M_{E_\alpha}\nu) \\
&\quad - \sum_{\alpha=p+1}^{n} g_L(\pi^\perp(\nabla_{E_\alpha}\nu), \pi^\perp(\nabla_{E_\alpha}\nu)) \\
&= g_M(\Delta_M\nu, \nu) - \sum_{i=1}^{p} g_L(W(\nu)E_i, W(\nu)E_i) \\
&\quad - \sum_{i=1}^{p} g_Q(A_\nu E_i, A_\nu E_i) - \sum_{\alpha=p+1}^{n} g_L(A_{E_\alpha}\nu, A_{E_\alpha}\nu) \\
&= g_M(\Delta_M\nu, \nu) - \| W(\nu) \|^2 - \| A(\nu) \|^2 .
\end{aligned}$$

In the above caculations, we have used the fact that $W(\nu)E_\alpha = 0$ and $A_{E_\alpha}\nu = -A_\nu E_\alpha$ for $\alpha = p+1, \cdots, n$. □

If there is an $\epsilon > 0$ such that for all $|t| < \epsilon$, $E(\mathcal{F}_t) \geq E(\mathcal{F})$, then the foliation $\mathcal{F}$ is said to be a *stable*.

Corollary 3.4. *Let $\mathcal{F}$ be a harmonic Riemannian foliation on a compact oriented Riemannian manifold M with a bundle-like metric g_M. If $\| W \|^2 + \| A \|^2 \leq -\rho(z)$ for any $z \in L^\perp$, where $\rho(z)$ is the partial Ricci curvature in the direction of z, then $\mathcal{F}$ is stable.*

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References

1. James Eells Jr. and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160.
2. Franz W. Kamber and Philppe Tondeur, *Harmonic foliations*, Sprin- ger lecture notes in mathematics **949** (1982), 87–121.
3. Franz W. Kamber and Philppe Tondeur, *The index of harmonic foliations on spheres*, Transactions of the American Mathematical Society **279**; 1 (1983), 257–263.
4. Franz W. Kamber and Philppe Tondeur, *Infinitesimal automorphisms and second variarion of the energy for harmonic foliations*, Tôhoku Math. Journ. **34** (1982), 525–538.
5. Barrett O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459–469.
6. Philppe Tondeur, Foliations on Riemannian manifold, *Lecture note* (1986).

Department of Mathematics, Yonsei University, Seoul 120-749, Korea
E-mail address: kimhb@yonsei.ac.kr