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SPECTRAL INVARIANTS OF ALMOST ISOSPECTRAL RIEMANNIAN FOLIATIONS

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Abstract. Main interest of the present paper is to consider spectral invariants of the transversal Jacobi operator for Riemannian foliations. Under some almost isospectral condition we obtain certain spectral determinations for the integrability of the normal bundle and harmonicity for the foliation. Our results extend some results in [7].

1. Introduction

Inverse spectral geometry has been intensively studied by many mathematicians since M. Kac ([5]). It has close relationships with diverse areas such as global and local differential geometry, algebraic topology, analysis of pseudodifferential operators on manifolds and mathematical physics.

The basic problem of this topic is to consider what geometric or topological properties of a manifold are determined or not determined from spectral data of a bounded from below elliptic pseudodifferential operator P, say the Laplacian of a Riemannian manifold. The general spectrum of P on a even simple domain are rarely known. Inverse spectral geometry therefore essentially treats with various forms of asymptotic expansion.

Let (M, g) and $(\tilde{M}, \tilde{g})$ be two oriented closed Riemannian manifolds. The Laplacians Δ and $\tilde{\Delta}$ acting on functions have discrete spectra $\mathrm{spec}(M, \Delta)$ and $\mathrm{spec}(\tilde{M}, \tilde{\Delta})$ with finite multuplicities respectively. The basic problem is then that if M and $\tilde{M}$ are isospectral, that is, $\mathrm{spec}(M, \Delta) = \mathrm{spec}(\tilde{M}, \tilde{\Delta})$ then (M, g) and $(\tilde{M}, \tilde{g})$ are isometric.

From the point of view of the transversal geometry Nishikawa, Tondeur and Vanhecke ([5]) considered an analogous problem what geometric properties of a Riemannian foliation $\mathcal{F}$ on an oriented closed Riemannian manifold (M, g) are determined or not determined from isospectral data of the Laplacian Δ and the Jacobi operator $\bar{J}_D$. Spectral invariants considered there are mainly the integrability of the normal bundle Q and the transversal curvature properties of $\mathcal{F}$.

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On the other hand, Wu ([10]) introduced a weaker notion than isospectrality, so called, α-isospectrality. Under the almost isospectral condition he showed several spectral results.

In the present paper we are interested in the spectral invariants of the transversal Jacobi operator J_D for Riemannian foliations. J_D is defined in terms of $\bar{J}_D$ and the tension field τ for $\mathcal{F}$ ([8]). By virtue of this operator we spectrally determine the harmonicity ($\tau = 0$) for the foliation under almost isospectral condition. Moreover we extend certain spectral determinations for the integrability of the normal bundle obtained in [7].

2. The transversal Jacobi operator

Let $(M, g, \mathcal{F})$ be a m-dimensional oriented closed Riemannian manifold with Riemannian foliation $\mathcal{F}$ of codimension $q := m - p$ and a bundle-like metric g. It is given by an exact sequence of vector bundles

$$0 \longrightarrow \mathcal{V} \longrightarrow TM \xrightarrow{\pi} Q \longrightarrow 0,$$

where $\mathcal{V}$ is the tangent bundle and Q the normal bundle of $\mathcal{F}$. The metric g determines an orthogonal decomposition $TM = \mathcal{V} \oplus \mathcal{H}$. We identify $\mathcal{H}$ with Q by an isometric splitting

$$(Q, g_Q) \cong (\mathcal{H}, g_{\mathcal{H}}). \tag{2.1}$$

We have an associated exact sequence of Lie algebras

$$0 \longrightarrow \Gamma(\mathcal{V}) \longrightarrow V(\mathcal{F}) \xrightarrow{\pi} \bar{V}(\mathcal{F}) \longrightarrow 0,$$

where $V(\mathcal{F}) := \{Y \in \Gamma(TM) | [V, Y] \in \Gamma(\mathcal{V}) \text{ for all } V \in \Gamma(\mathcal{V})\}$ and $\bar{V}(\mathcal{F}) := \{s \in \Gamma(Q) | s = \pi(Y), Y \in V(\mathcal{F})\}$. We denote by $\Gamma(\cdot)$ the space of all smooth sections of a vector bundle $(\cdot)$. The transversal Levi-Civita connection D on Q is a torsion free and metric connection with respect to g_Q ([7], [9]).

Throughout this paper, we use the following notations.

τ : the tension field of $\mathcal{F}$,
$\mathrm{grad}_D\, f$: the transversal gradient of a function f,
$\mathrm{div}_D\, s$: the transversal divergence of $s \in \Gamma(Q)$,
R_D : the transversal curvature tensor,
ρ_D : the transversal Ricci operator,
s_D : the transversal scalar curvature,
$\Delta_D = \mathrm{grad}_D \mathrm{div}_D$: the Laplacian acting on $\Gamma(Q)$,
$\theta(Y)$: the transversal Lie derivative operator for $Y \in V(\mathcal{F})$,
$A_D(Y) := \theta(Y) - D_Y$ for $Y \in V(\mathcal{F})$.

It is noted that the notions $\theta(s)$ and $A_D(s)$ make sense for a given $s \in \bar{V}(\mathcal{F})$.

The basic complex $(\Omega_B, d_B := d|_{\Omega_B})$ is a subcomplex of the de Rham complex $(\Omega(M), d)$, where

$$\Omega_B := \{\omega \in \Omega(M) | i_V\omega = \theta(V)\omega = 0 \text{ for all } V \in \Gamma(\mathcal{V})\}.$$

We often use the following identification by means of (2.1) and g_Q−duality

$$\bar{V}(\mathcal{F}) \cong \Omega_B^1. \tag{2.2}$$

In what follows, we assume that $\tau \in \bar{V}(\mathcal{F})$, which is based on the result given by Domínguez ([3])

Theorem 2.1. *Let $\mathcal{F}$ be Riemannian foliation on a closed manifold M. Then $\mathcal{F}$ is tense, i.e., there is a bundle-like metric g on M satisfying $\tau \in \bar{V}(\mathcal{F})$.*

The Jacobi operator of $\mathcal{F}$ on $\Gamma(Q)$ is given by

$$\bar{J}_D := \Delta_D - \rho_D.$$

Then the transversal Jacobi operator appeared in [8] is defined by

$$J_D := \bar{J}_D - A_D(\tau).$$

This operator follows from special second variation treated in [6].

Proposition 2.2. *Let $(M, g, \mathcal{F})$ be a closed Riemannian manifold with Riemannian foliation $\mathcal{F}$ and a bundle-like metric g. If $\tau \in \bar{V}(\mathcal{F})$ then J_D is symmetric and preserves $\bar{V}(\mathcal{F})$.*

Proof. Note that $\tau \in \bar{V}(\mathcal{F})$ means $\kappa \in \Omega_B^1$. Here κ is the mean curvature form for $\mathcal{F}$ which is denoted by the g_Q-dual to τ in the sense of (2.2). It follows that $d\kappa = 0$ ([9]). This is equivalent that $A_D(\tau)$ is symmetric. Therefore, J_D is symmetric. It is clear that J_D preserves $\bar{V}(\mathcal{F})$.

With respect to the natural scalar product on $\Gamma(Q)$ J_D is elliptic of the second order with leading symbol g. So it has a discrete spectrum with finite multiplicities.

Recall the heat invariants $a_k(\Delta)$ and $b_k(J_D)$ defined by the asymptotics of the heat kernel ([1], [4])

$$\mathrm{Tr}(e^{-\Delta t}) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \overset{t\downarrow 0}{\sim} (4\pi t)^{-m/2} \sum_{k=0}^{\infty} a_k(\Delta) t^k,$$

$$\mathrm{Tr}(e^{-J_D t}) = \sum_{i=0}^{\infty} e^{-\mu_i t} \overset{t\downarrow 0}{\sim} (4\pi t)^{-m/2} \sum_{k=0}^{\infty} b_k(J_D) t^k.$$

They are spectral invariants depending only on the discrete spectra

$$\mathrm{spec}(M, \Delta) = \{0 = \lambda_0 < \lambda_1 \le \cdots \le \lambda_i \nearrow \infty\},$$

$$\mathrm{spec}(M, J_D) = \{\mu_0 \le \mu_1 \le \cdots \le \mu_i \nearrow \infty\}$$

respectively. The curvature data associated to (M, g) are denoted by R, ρ and s. Then we have the formulas for $a_k(\Delta)$ and $b_k(J_D)$ (cf. [1], [4]).

Theorem 2.3. *Let $(M, g, \mathcal{F})$ be a m-dimensional oriented closed Riemannian manifold with Riemannian foliation $\mathcal{F}$ of codimension $q := m - p$ and a bundle-like metric g satisfying $\tau \in \bar{V}(\mathcal{F})$. Then*

$$\begin{aligned}
a_0(\Delta) &= vol(M),\\
a_1(\Delta) &= \frac{1}{6}\int_M s\mu,\\
a_2(\Delta) &= \frac{1}{360}\int_M [2|R|^2 - 2|\rho|^2 + 5s^2]\mu,\\
b_0(J_D) &= qa_0(\Delta),\\
b_1(J_D) &= qa_1(\Delta) + \int_M [s_D - |\tau|^2]\mu,\\
b_2(J_D) &= qa_2(\Delta) + \frac{1}{12}\int_M [2s(s_D - div_D\ \tau)\\
&\quad + 6(|\rho_D|^2 + 2g_Q(\rho_D, A_D(\tau)) + |D\tau|^2) - |R_D|^2]\mu,
\end{aligned}$$

where μ denotes the Riemannian volume form on (M, g).

3. Almost isospectral Riemannian foliations.

Let $(M, g, \mathcal{F})$ and $(\tilde{M}, \tilde{g}, \tilde{\mathcal{F}})$ be m-dimensional oriented closed Riemannian manifolds with Riemannian foliations $\mathcal{F}$ and $\tilde{\mathcal{F}})$ of codimension $q := m - p$ and a bundle-like metric g and $\tilde{g}$ satisfying $\tau \in \bar{V}(\mathcal{F})$ and $\tilde{\tau} \in \bar{V}(\tilde{\mathcal{F}})$ respectively. We say that two Riemannian foliations $(M, g, \mathcal{F})$ and $(\tilde{M}, \tilde{g}, \tilde{\mathcal{F}})$ are isospectral if $\mathrm{spec}(M, \Delta) = \mathrm{spec}(\tilde{M}, \Delta_{\tilde{g}})$ and $\mathrm{spec}(M, J_D) = \mathrm{spec}(\tilde{M}, \tilde{J}_D)$. More generally, they are α-isospectral for a real number α if

$$\limsup_{n\to\infty} \frac{|\lambda_n - \tilde{\lambda}_n|}{n^\alpha} < \infty \quad \text{and} \quad \limsup_{n\to\infty} \frac{|\mu_n - \tilde{\mu}_n|}{n^\alpha} < \infty.$$

It is obvious that isospectral Riemannian foliations are α-isospectral for all α. However, the converse does not hold.

Consider the case of codimension one foliation. The two 2-dimensional non-isospectral flat tori given in [10] carry trivial Riemannian flows obtained by the projecting the lattices onto the line generated by the first of two independent basic vectors of the lattice. In this case, the sections of the normal bundle can be identified with functions on the manifold. Since the Riemannian flows are both harmonic, each transversal Jacobi operator coincides with the Laplacian acting on functions. These two flat tori are 0-isospectral, but not isospectral. Thus we have an example of 0-isospectral Riemannian foliations which are not isospectral.

Under almost isospectral condition we have the following result. The proof is similar as in [10].

Proposition 3.1. *Let $(M, g, \mathcal{F})$ and $(\tilde{M}, \tilde{g}, \tilde{\mathcal{F}})$ be α-isospectral Riemannian foliations. Suppose that both M and $\tilde{M}$ have same dimension m. If $\alpha = -1$ then $a_k(\Delta) = a_k(\tilde{\Delta})$ and $b_k(J_D) = b_k(\tilde{J}_D)$ for all $k \leq [\frac{m+1}{2}]$.*

From Theorem 2.3 and Proposition 3.1 we have

Corollary 3.2. *Let $(M, g, \mathcal{F})$ and $(\tilde{M}, \tilde{g}, \tilde{\mathcal{F}})$ be (-1)-isospectral Riemannian foliations. Then we have*

1. $\dim M = \dim \tilde{M}$,
2. $\mathrm{vol}(M) = = \mathrm{vol}(\tilde{M})$,
3. $\int_M s\mu = \int_{\tilde{M}} \tilde{s}\tilde{\mu}$,
4. $\mathrm{codim}\ \mathcal{F} = \mathrm{codim}\ \tilde{\mathcal{F}}$,
5. $\int_M [s_D - |\tau|^2]\mu = \int_{\tilde{M}} [\tilde{s_D} - |\tilde{\tau}|^2]\tilde{\mu}$,
6. $\int_M [2|R|^2 - 2|\rho|^2 + 5s^2]\mu = \int_{\tilde{M}} [2|\tilde{R}|^2 - 2|\tilde{\rho}|^2 + 5\tilde{s}^2]\tilde{\mu}$,
7.

$$\int_M [2s(s_D - div_D\ \tau) + 6(|\rho_D|^2 + 2g_Q(\rho_D, A_D(\tau)) + |D\tau|^2) - |R_D|^2]\mu =$$

$$\int_{\tilde{M}} [2\tilde{s}(\tilde{s}_D - div_{\tilde{D}}\ \tilde{\tau}) + 6(|\tilde{\rho}_D|^2 + 2\tilde{g}_Q(\tilde{\rho}_D, \tilde{A}_D(\tilde{\tau})) + |\tilde{D}\tilde{\tau}|^2) - |\tilde{R}_D|^2]\tilde{\mu}.$$

4. Almost isospectral invariants.

We apply Corollary 3.2 to obtain spectral invariant of almost isospectral Riemannian folaitions. In particular case of Riemannian flows, we have

Theorem 4.1. *Let $(M, g, \mathcal{F})$ and $(\tilde{M}, \tilde{g}, \tilde{\mathcal{F}})$ be (-1)-isospectral Riemannian flows on Einstein manifolds. Then*

1. *the normal bundle Q is integrable if and only if $\tilde{Q}$ is integrable,*
2. *$\mathcal{F}$ is harmonic if and only if $\tilde{\mathcal{F}}$ is harmonic.*

Proof. We need O'Neill structure tensors T and A ([2], [9]) for $\mathcal{F}$. Take an orthonormal frame $\{V, E_a\}$ on M such that $\pi(E_a) \in \bar{V}(\mathcal{F})$ ([8]). Since M and $\tilde{M}$ are Einstein, it can be written as $\rho = c$ id and $\tilde{\rho} = \tilde{c}$ id. Then Corollary 3.2 (3) implies $c = \tilde{c}$.

On the other hand, we have the Ricci curvature relation.

$$\begin{aligned}\rho(V, V) &= -|\tau|^2 + |A|^2 + \sum g((\nabla_{E_a})T_V V, E_a) \\ &= |A|^2 + \mathrm{div}\ \tau,\end{aligned}$$

where ∇ denotes the Levi-Civita connection with respect to g. It follows that

$$\int_M |A|^2\mu = c\mathrm{vol}(M).$$

Thus we see

$$\int_M |A|^2\mu = \int_{\tilde M} |\tilde A|^2\tilde\mu. \tag{4.1}$$

This proves (1). Moreover, the Ricci curvature relation

$$\sum \rho(E_a, E_a) = s_D - |T|^2 - 2|A|^2 + \sum g((\nabla_{E_a}\tau, E_a) \tag{4.2}$$

gives rise to

$$\int_M [s_D - 2|A|^2]\mu = cq\mathrm{vol}(M),$$

so that

$$\int_M [s_D - 2|A|^2]\mu = \int_{\tilde M} [\tilde s_D - 2|\tilde A|^2]\tilde\mu.$$

This, together with (4.1) yileds

$$\int_M s_D\mu = \int_{\tilde M} \tilde s_D\tilde\mu. \tag{4.3}$$

Therefore Corollary 3.2 (5) and (4.3) deduce

$$\int_M |\tau|^2\mu = \int_{\tilde M} |\tilde\tau|^2\tilde\mu,$$

which proves (2).

Theorem 4.2. *Let* $(M, g, \mathcal{F})$ *and* $(\tilde M, \tilde g, \tilde{\mathcal{F}})$ *be* (-1)*-isospectral totally geodesic Riemannian foliations on Einstein manifolds. Then* Q *is integrable if and only if* $\tilde Q$ *is integrable.*

Proof. ¿From the totally geodesic condition, we have from (4.2)

$$\frac{q}{m}s = s_D - 2|A|^2. \tag{4.4}$$

(4.4) and Corollary 3.2 imply (4.1). Therefore we have the conclusion.

Theorem 4.2 (1) and Theorem 4.3 extend the results established in [7]. Furthermore, when the foliations considered are harmonic we have similar spectral invariants for the transversal curvature properties under almost isospectral condition.

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