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COMPACT EINSTEIN WARPED PRODUCT SPACES

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Abstract. We study Einstein warped product spaces. As a result, we prove the following: there does not exist a compact Einstein warped product space with nonconstant warping function, if the scalar curvature is nonpositive or the base is of 2-dimensional.

0. Introduction

Let $B = (B^m, g_B)$ and $F = (F^k, g_F)$ be two Riemannian manifolds. We denote by π and σ the projections of $B \times F$ onto B and F, respectively. For a positive smooth function f on B the warped product $M = B \times_f F$ is the product $M = B \times F$ furnished with metric tensor g defined by $g = \pi^* g_B + f^2 \sigma^* g_F$, where (*) denotes pull back. The function f will be called the warping function.

The notion of warped product $B \times_f F$ generalizes that of a surface of revolution. It was introduced in [3] for studying manifolds of negative curvature.

Obviously, the Riemannian product $M = B \times F$ is Einstein if B and F are Einstein with the same scalar curvarures. A warped product $B \times_f F$ with a constant warping function f can be considered as a Riemannian product.

In search of a new compact Einstein space in ([2], p. 265), A. L. Besse asked the following:

"Does there exist a compact Einstein warped product with nonconstant warping function? "

In this talk, we give some negative partial answers as follows([6], [8]):

Theorem 1. *Let $M = B \times_f F$ be a compact Einstein warped product space. If M has nonpositive scalar curvature, then the warped product becomes a Riemannian product.*

Theorem 2. *Let $M = B \times_f F$ be a compact Einstein warped product space. If the base B is of 2-dimensional, then the warped product becomes a Riemannian product.*

This is a survey article on the compact Einstein warped product spaces. For a detailed proof, see [6] and [8].

1. Einstein warped product spaces with nonpositive scalar curvature

We denote by $\mathrm{Ric}^B, \mathrm{Ric}^F$ the lifts to M of the Ricci curvatures of B and F, respectively. Then we have the following ([10]):

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Proposition 3. *The Ricci curvature Ric of the warped product $M = B \times_f F$ with $k = \dim F$ satisfies*
(1) $\mathrm{Ric}(X,Y) = \mathrm{Ric}^B(X,Y) - \frac{k}{f} H^f(X,Y)$,
(2) $\mathrm{Ric}(X,V) = 0$,
(3) $\mathrm{Ric}(V,W) = \mathrm{Ric}^F(V,W) - g(V,W)f^{\#}$, $f^{\#} = \frac{-\Delta f}{f} + \frac{k-1}{f^2} g_B(\nabla f, \nabla f)$ *for any horizontal vectors X, Y and any vertical vectors V, W, where H^f and Δf denote the Hessian of f and the Laplacian of f given by $-tr(H^f)$, respectively.*

Hence the Einstein equations become

Corollary 4. *The warped product $M = B \times_f F$ is Einstein with $\mathrm{Ric} = \lambda g$ if and only if*
(1.1) $\mathrm{Ric}_B = \lambda g_B + \frac{k}{f} H^f$,
(1.2) (F, g_F) *is Einstein with* $\mathrm{Ric}_F = \mu g_F$,
(1.3) $-f\Delta f + (k-1)|\nabla f|^2 + \lambda f^2 = \mu$.

Now we state a lemma. For a detailed proof, see [8].

Lemma 5. *Let f be a smooth function on a Riemannian manifold B, then for any vector X, the divergence of the Hessian tensor H^f satisfies*

$$\mathrm{div}(H^f)(X) = \mathrm{Ric}(\nabla f, X) - \Delta(df)(X), \tag{1.4}$$

where $\Delta = d\delta + \delta d$ denotes the Laplacian on B acting on differential forms.

Using the above lemma, we may prove the following proposition:

Proposition 6. *Let (B^m, g_B) be a compact Riemannian manifold of dimension $m \geqq 2$. Suppose that f is a nonconstant smooth function on B satisfying* (1.1) *for a constant $\lambda \in R$ and a natural number $k \in N$. Then f satisfies* (1.3) *for a constant $\mu \in R$. Hence for a compact Einstein space (F, g_F) of dimension k with $\mathrm{Ric}_F = \mu g_F$, we can make a compact Einstein warped product space $M = B \times_f F$ with $\mathrm{Ric} = \lambda g$.*

Proof. For a proof, see [8].

Now we give the proof of Theorem 1. Note that (1.3) becomes

$$\mathrm{div}(f\nabla f) + (k-2)|\nabla f|^2 + \lambda f^2 = \mu. \tag{1.5}$$

By integrating (1.5) over B we have

$$\mu = \frac{k-2}{V(B)} \int_B |\nabla f|^2 + \frac{\lambda}{V(B)} \int_B f^2, \tag{1.6}$$

where $V(B)$ denotes the volume of B.

1) Suppose $k \geq 3$. Let p be a maximum point of f on B. Then, we have $f(p) > 0, \nabla f(p) = 0$ and $\Delta f(p) \geq 0$. Hence from (1.3) and (1.6) we obtain the following:

$$\begin{aligned} 0 &\leqq f(p)\Delta f(p) \\ &= \lambda f(p)^2 - \mu \\ &= \frac{2-k}{V(B)} \int_B |\nabla f|^2 + \frac{\lambda}{V(B)} \int_B (f(p)^2 - f^2) \\ &\leqq 0. \end{aligned}$$

The last inequality follows from the hypothesis on λ. Thus, f is constant.

2) When $k = 1, 2$, we choose q as a minimum point of f on B. Then, we have $f(q) > 0, \nabla f(q) = 0$ and $\Delta f(q) \leq 0$. Hence we obtain from (1.3) and (1.6)

$$
\begin{aligned}
0 &\geqq f(q)\Delta f(q) \\
&= \lambda f(q)^2 - \mu \\
&= \frac{2-k}{V(B)} \int_B |\nabla f|^2 + \frac{\lambda}{V(B)} \int_B (f(q)^2 - f^2) \\
&\geqq 0.
\end{aligned} \tag{1.7}
$$

As is in case 1, the last inequality follows from the hypothesis on λ. If $k = 1$ or $\lambda < 0$, then (1.7) shows that f is constant. If $k = 2$ and $\lambda = 0$, (1.5) and (1.6) imply that f^2 is harmonic on B, and hence f is constant. This completes the proof of the theorem.

In a similar manner, we may prove the following (cf. [4]):

Remark 7. *Let (M, g) be a compact Riemannian manifold. If the Ricci tensor satisfies* $\mathrm{Ric} = \lambda g + H^f$ *for a nonpositive constant $\lambda \in R$ and a smooth function f on M, then f is constant.*

2. Einstein warped product spaces with 2-dimensional base

In this section, we give a proof of Theorem 2. Let (B, g_B) be a compact 2-dimensional Riemannian surface. By Theorem 1, we may assume that $\lambda > 0$. Since B is of 2-dimensional, the Ricci tensor satisfies $\mathrm{Ric}_B = K g_B$, where K denotes the Gaussian curvature of B. Hence (1.1) becomes

$$H^f = \frac{f}{k}(K - \lambda) g_B. \tag{2.1}$$

Suppose that the warping function f is nonconstant. Then (2.1) shows that if p, q denote the minimum and maximum points of f, then $(B - \{p, q\}, g_B)$ is isometric with a warped product metric(Theorem 21 of [9])

$$ds^2 = dt^2 + f'(t)^2 d\theta^2 \tag{2.2}$$

on $(a, b) \times S^1$, where $f = f(t)$ and $f'(t) = \frac{df}{dt}$. Obviously, we have

$$f'(a) = f'(b) = 0. \tag{2.3}$$

Since the metric (2.2) extends to a C^∞ Riemannian metric on B, we may assume that ([2], p.269 or [9], p. 123)

$$f''(a) = -f''(b) = 1. \tag{2.4}$$

Note that $\Delta f = -2f''(t)$ in the metric (2.2). Hence (1.3) becomes

$$2f(t)f''(t) + (k-1)f'(t)^2 + \lambda f(t)^2 = \mu. \tag{2.5}$$

Hereafter, we assume that the dimension k of the fibre F is greater than or equal to 2. Integrating (2.5), we get

$$f'(t)^2 = \frac{\mu}{k-1} - \frac{\lambda}{k+1} f(t)^2 + \nu f(t)^{1-k}, \tag{2.6}$$

and hence

$$f''(t) = -\frac{\lambda}{k+1} f(t) - \frac{k-1}{2} \nu f(t)^{-k}, \tag{2.7}$$

where ν is a constant.

Now if we put

$$\begin{aligned} g(x) &= \frac{\mu}{k-1} - \frac{\lambda}{k+1} x^2 + \nu x^{1-k} \\ &= x^{1-k}\left(-\frac{\lambda}{k+1} x^{k+1} + \frac{\mu}{k-1} x^{k-1} + \nu\right), \end{aligned} \tag{2.8}$$

then we have $f'(t)^2 = g(f(t))$ and $f''(t) = \frac{1}{2} g'(f(t))$. If A, B denote the minimum $f(a) = f(p)$ and maximum $f(b) = f(q)$ of f, respectively, then (2.3) and (2.4) imply

$$g(A) = 0, \quad g'(A) = 2, \tag{2.9}$$

and

$$g(B) = 0, \quad g'(B) = -2. \tag{2.10}$$

¿From (2.8) and (2.9) we get

$$\nu = \frac{-2}{k^2-1}(\sqrt{1+\mu\lambda} + k) A^k, \tag{2.11}$$

and

$$A = \frac{1}{\lambda}(\sqrt{1+\mu\lambda} - 1). \tag{2.12}$$

Similarly, from (2.8) and (2.10) we obtain

$$B = \frac{1}{\lambda}(\sqrt{1+\mu\lambda} + 1). \tag{2.13}$$

Since $g(B) = 0$, from (2.8), (2.11), (2.12) and (2.13) we see that the positive constant $y = \sqrt{1+\mu\lambda}$ is a positive zero of the following polynomial:

$$h_k(y) = (k-1)(y+1)^{k+1} - (k+1)(y^2-1)(y+1)^{k-1} + 2(y+k)(y-1)^k. \tag{2.14}$$

It can be easily shown that $h_k(y)$ is a polynomial of degree $k-2$ which can be expanded as follows:

$$h_k(y) = 8 \sum_{j=1}^{[\frac{k-1}{2}]} j \binom{k+1}{2j+1} y^{k-2j},$$

where [.] denotes the Gaussian integer function. Since all the coefficients of $h_k(y)$ are positive, it cannot have a positive zero. This contradiction completes the proof of Theorem 1 in case $k \geq 2$. If $k = 1$, then a similar argument to the above proves the theorem.

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