Trends in Mathematics
Information Center for Mathematical Sciences
Volume 5, Number 2,December 2002, Pages 115–124

STABLY EMBEDDED MINIMAL HYPERSURFACES

BARBARA NELLI & MARC SORET

ABSTRACT. We use Schoen's estimates to prove that the subfocal tubular neighborhood of a nonflat minimal hypersurface stably embedded in $R^{n+1}, n \leq 5$, can not be embedded. In particular such hypersurfaces, with bounded curvature admit no embedded tubular neighborhoods of constant radius, whatever small the radius. On the contrary, we prove that such hypersurfaces, unless they are everywhere dense, admit an embedded tube whose radius decays sufficiently fast. In particular they are proper.

0. INTRODUCTION

It is known that a complete stable minimal surface in $\mathbb{R}^3$ is a plane [CP]. On the contrary, there are examples of stable minimal hypersurfaces embedded in $\mathbb{R}^{n+1}$ for $n \geq 7$ [BGG]. It is an open problem to know if the only stable hypersurfaces in $\mathbb{R}^{n+1}, 3 \leq n \leq 6$ is a hyperplane. We give new results for embedded stable minimal hypersurfaces.

Let us recall known results for hypersurfaces in $\mathbb{R}^{n+1}, 3 \leq n \leq 6$.

If the L^p norm of the second fundamental form of a hypersurface M of $\mathbb{R}^{n+1}$ is bounded for some $p \in]0, 4 + \sqrt{\frac{8}{n}}[$ then M is a hyperplane [SSY]. This implies in particular that there is no minimal graph. Furthermore in [CP] it is proved that a minimal hypersurface is a hyperplane as soon as

$$\lim_{R \longrightarrow \infty} \frac{\int_{B_R} |K|}{R^{2+2q}} = 0, \quad q < \sqrt{2/n}.$$

If M has more than one end then M is a hyperplane [CSZ]. This result has been extended recently to any ambient space with positive sectional curvature [LW] . In [C] it is showed that if the number of connected components of the intersection between M and any ball of $\mathbb{R}^{n+1}$, $n \leq 4$, is bounded by some constant, then M is an hyperplane.

The paper is organized as follows.

In Section 1, we rapidly show why the subfocal tubular neighborhood of an embedded non-flat stable minimal hypersurface has self-intersection.

In Section 2, we give without proof the equation that ϕ satisfies, where ϕ is the distance function from points of a minimal piece of a minimal hypersurface to another minimal piece. It turns out that this minimal equation has a simple form from which we deduce a linear inequality for ϕ.

2000 *Mathematics Subject Classification.* Primary 53 A 10.
Key words and phrases. minimal, hypersurfaces, stable.
Received October 1, 2002

In Section 3, in opposition to Section 1, we sketch the proof that, in general, embedded minimal hypersurfaces M with bounded geometry (that is the length of the second fundamental form of M is bounded) are proper, that is, there exists a tubular neighborhood of M that is embedded.

1. VOLUME OF TUBES

We will give a geometric application of an estimate found in [SSY]. Let M be a hypersurface in $\mathbb{R}^{n+1}$ with an orientation $\mathbf{N}$ and let $r : M \longrightarrow \mathbb{R}$ be a smooth positive function. Fix a point σ of $\mathbb{R}^{n+1}$ and let $B(\sigma, R)$ be the Euclidean ball of radius R centered at σ. Denote by B_R the intersection $B(\sigma, R) \cap M$. Consider a bundle $\pi : U \longrightarrow M$ with U an open neighborhood of M in $\mathbb{R}^{n+1}$, for which the zero section $s : M \longrightarrow U$ is the inclusion of M in U. We call such a bundle a *Tubular neighborhood* of M in $\mathbb{R}^{n+1}$ and denote $s(M) = M_0$. It is clear that one can choose a tubular neighborhood that is locally equivalent to a domain of the normal bundle; hence the exponential map $\exp : NM \longrightarrow \mathbb{R}^{n+1}$ is a local diffeomorphism between U and a neighborhood of the zero section of the normal bundle. We will often write our definitions up to the local diffeomorphism $\exp$. This diffeomorphism provides a mean to compute the volume of $T(M, r)$ in terms of the radius function r and coordinates on M. More generally any homothety of ratio less than one of the subfocal tube will still be named as subfocal.

Definition 1.1. We call *Tube* around M of radius r the set

$$T(M, r) = \{x \in \mathbb{R}^{n+1} \mid \exists p \in M,\ x = p + t\mathbf{N}(p),\ t \leq r(p)\}$$

Denote by $T(R, r)$ the tube of radius r around $B_R \subset M$, by dT its volume element and by $V(R, r)$ its volume *i.e.*

$$V(R, r) = \int_{T(R,r)} dT.$$

Theorem 1. *Let M be a stable minimal hypersurface in $\mathbb{R}^{n+1}$, $n \leq 5$ that is not a plane. Then, there exists R_0 such that for every $R > R_0$ the subfocal tube $T(R, r)$, $r(p) = c|A(p)|^{-1}$, for any $c \leq 1$, is not embedded.*

In order to prove the Theorem we need some lemmae. Let

$$M_t = \{x \in \mathbb{R}^{n+1} \mid \exists p \in M,\ x = p + t\mathbf{N}(p)\}$$

M_t is called parallel surface to M at a distance t. We note that

$$M_t = \{x \in \mathbb{R}^{n+1} \mid d(x, M) = t\}$$

and that $dT = dM_t dt$, where dM_t is the volume element of M_t. It is not hard to see that

$$dM_t = \det\,(1 - tA) dM$$

where A and dM are the second fundamental form and the volume element of M respectively (cf. [G] for details). By a straightforward computation we have:

$$\det\,(1 - tA) = \prod_{l=1}^{n}(1 - t\kappa_l) = 1 + \sum_{h=1}^{n}(-1)^h t^h S_h$$

where κ_h are the principal curvatures of M and S_h are the elementary symmetric functions of κ_h. Choose $r(p) \le c|A(p)|^{-1}$, where c is any constant $|A(p)|$ is the norm of the second fundamental form of M at the point p, in other words we consider a subfocal tubular neighborhood of M ($c \le 1$). Let us find an estimate from below for the volume of the subfocal tube around M.

$$V(R,r) = \int_{T(R,r)} dM_t dt = \int_{T(R,r)} (1 + \sum_{h=1}^{n} (-1)^h t^h S_h) dM dt$$

$$= \int_{B_R} dM \int_0^{r(p)} (1 + \sum_{h=1}^{n} (-1)^h t^h S_h) dt$$

From now on, we assume that the hypersurface M is minimal, hence $S_1 = 0$ and

$$V(R,r) = \int_{B_R} r(p) dM + \sum_{h=2}^{n} \frac{(-1)^h}{h+1} \int_{B_R} r(p)^{h+1} S_h dM$$

Now, a fundamental result of [SSY] will give estimates on the volume of B_R and on the L^p norm of the second fundamental form of M. We recall

Theorem[SSY] *Let M be a stable minimal hypersurface of $\mathbb{R}^{n+1}$ then, for any $p \in \left]0, 4 + \sqrt{\frac{8}{n}}\right[$*

$$\int_{B_R} |A|^p \le \beta(p,n) R^{-p} |B_R| \tag{1}$$

where $|B_R|$ is the volume of B_R and $\beta(p,n)$ is a constant that depends on n and p.

Lemma 1. *Let $n \le 6$. If M is a stable minimal non planar hypersurface in $\mathbb{R}^{n+1}$, then there exists a constant $\gamma > 0$ that only depends on n such that for every positive $\varepsilon < -\frac{1}{2} + \sqrt{\frac{2}{n}}$ and R sufficiently big,*

$$|B_R| > \gamma R^{5+\varepsilon} \tag{2}$$

Proof. As $n \le 7$, in (1) we can choose $p = 5 + 2\varepsilon$ and we have

$$\int_{B_R} |A|^{5+2\varepsilon} \le \beta R^{-5-2\varepsilon} |B_R|$$

provided $\varepsilon < -\frac{1}{2} + \sqrt{\frac{2}{n}}$. By contradiction, suppose that for any positive constant γ there exists some R arbitrarily big such that

$$|B_R| \le \gamma R^{5+\varepsilon}$$

It follows that

$$\int_{B_R} |A|^{5+2\varepsilon} \le \beta\gamma R^{-5-2\varepsilon} R^{5+\varepsilon} = \frac{\beta\gamma}{R^{\varepsilon}}$$

Let R go to infinity, we deduce that $|A| \equiv 0$ on M, hence M is a hyperplane : contradiction.

In what follows we assume that for all R large enough and for all $\varepsilon < -\frac{1}{2} + \sqrt{\frac{2}{n}}$

$$|B_R| > \gamma R^{5+\varepsilon},$$

and take as the radius of the tube around M $r(p) = \frac{c}{|A|}, c = 1$ Then, by Schwarz inequality, it is easy to prove :

Lemma 2.

$$\int_{B_R} \frac{1}{|A|} \geq \beta(1,n)^{-2}|B_R|R$$

for a constant β that depends only on n.

Idea of the proof of Theorem 1. For $h = 0, \ldots, n$ let

$$V_h(R,r) = \frac{(-1)^h}{h+1}\int_{B_R} \frac{1}{|A|^{h+1}} S_h dM$$

be the term of order h of the volume of the tube. Denote by $S_{h,i}$ the h-th order elementary symmetric function of the $n-1$ variables κ_j, $j \in \{1,\ldots,n\} \setminus i$. We will use the following property of the function S_h (cf. [LT]):

$$hS_h = \sum_{i=1}^{n} S_{h-1,i}\kappa_i$$

Using previous equality, we obtain :

$$S_2 = \frac{|A|^2}{2}$$

$$S_3 = \sum_{i=1}^{n} \kappa_i^3$$

$$S_4 = \frac{1}{4!}(|A|^4 - \sum_{i=1}^{n} \kappa_i^4)$$

$$S_5 = \frac{1}{5!}(\sum_{i=1}^{n} \kappa_i^5 - 2|A|^2 \sum_{i=1}^{n} \kappa_i^3)$$

Furthermore, by (5) in next section and for $i = 1, \ldots, n$, we have

$$|\kappa_i| \leq \sqrt{\frac{n-1}{n}}|A|.$$

Then, it is not hard to see that for every $h \leq n$, $n \leq 5$ there exist a constant $c(h,n)$ such that

$$(-1)^{h+1} S_h \leq c(h)|A|^h \tag{3}$$

and

$$\sum_{h=0}^{n} c(h,n) > 0.$$

We can choose $c(0,n)=1$, $c(1,n)=0$ and $c(2,n)=\frac{1}{2}$, that is

$$V_0(R,r)=\int_{B_R}\frac{1}{|A|}dM,$$

$$V_1(R,r)=0,$$

$$V_2(R,r)=-\frac{1}{6}\int_{B_R}\frac{1}{|A|}dM$$

Hence

$$V(R,r)\geq\sum_{h=0}^{n}\int_{B_R}\frac{c(h,n)}{|A|}dM=\left(1-\sum_{h=1}^{n}c(h,n)\right)\int_{B_R}\frac{1}{|A|}dM$$

$$\geq\left(1-\sum_{h=1}^{n}c(h,n)\right)|B_R|\beta(1,n)^{-2}R\geq CR^{6+\epsilon}$$

where C is a positive constant depending on n. But this contradicts lemma 1.4; consequently the tube $T(R,r)$ is not embedded.

From this theorem we prove :

Corollary 1. *Let M be a stable minimal hypersurface in $\mathbb{R}^{n+1}$, $n\leq 5$ that is not a plane; for any p in M, denote by c a positive constant and by $d(p,\sigma)$ the Euclidean distance between p and the origin; let us consider the function on M defined by $r(p)=\inf(|A(p)|^{-1},c.d(p,\sigma)^{-\delta})$. Let $\delta<\frac{7}{2}+\sqrt{\frac{2}{n}}-n$, then, the tube $T(M,r(p))$ is not embedded.*

Proof. The proof is similar to the proof of Theorem 1. In this case

$$V(R,r)=\int_{B_R}r(p)dM-\frac{1}{4}\int_{B_R}r(p)^4S_3dM$$

Denote by B_R^+ the subset of B_R where $r(p)=|A(p)|^{-1}$ and let $B_R^-=B_R\setminus B_R^+$. Then

$$V(R,r)=\int_{B_R^+}|A(p)|^{-1}dM-\frac{1}{4}\&\int_{B_R^+}|A(p)|^{-1}S_3dM+$$
$$\&\int_{B_R^-}d(p,\sigma)^{-\delta}dM-\frac{1}{4}\int_{B_R^-}d(p,\sigma)^{-4\delta}S_3dM$$

For the two integrals over B_R^+ we proceed as in the proof of Theorem 2.5. Let us evaluate the integrals over B_R^-. By Cauchy-Schwarz inequality we have

$$\int_{B_R^-}d(p,\sigma)^{-\delta}dM\geq R^{-\delta}|B_R^-|$$

Inequality (3) on S_3, implies that

$$-\int_{B_R^-}d(p,\sigma)^{-4\delta}S_3dM\geq-\frac{1}{6}\left(\frac{n-1}{n}\right)^{\frac{3}{2}}\int_{B_R^-}d(p,\sigma)^{-\delta}dM.$$

Hence there exists a constant $C(n)$ such that

$$V(R,r) \geq C(n)(|B_R + |R + |B_R^-|R^{-\delta}),$$

and

$$V(R,r) \geq C(n)\gamma R^{5+\varepsilon} \min(R, R^{-\delta}),$$

for $\varepsilon < -\frac{1}{2} + \sqrt{\frac{2}{n}}$. By the choice of δ we have

$$V(R,r) > C(n)\gamma R^{n+1}.$$

Consequently the tube $T(R,r)$ is not embedded.

Behaviour of non-embedded tubes.

Let M be a hypersurface of $\mathbb{R}^{n+1}$ with bounded curvature and let $T(M,r)$ be a subfocal tube around M.

(i) Assume that:

$$V(R,r) > R^{n+1}\omega_{n+1}$$

where ω_{n+1} is the volume of the unit ball in $\mathbb{R}^{n+1}$. Necessarily $T(M,r)$ has self-intersections since some points of $T(R,r)$ are taken at least twice for every R.

Note that we can choose a subfocal tube with constant radius $C \leq (\sup|A|)^{-1}$.

(ii) Assume M is minimal. Choose a subfocal tube with constant radius $C \leq (\sup|A|)^{-1}$, and suppose that

$$V(R, C/2) > R^{n+1}\omega_{n+1}$$

then M intersects the boundary of $T(M,C)$. More precisely, as we noticed at the beginning of Section 1, the tube $T(M,C)$ defines a neighborhood of the zero section M_0 of the normal bundle by the exponential map

$$\exp : NM \longrightarrow \mathbb{R}^{n+1}$$

and

$$\partial T(M,C) \cap \exp^{-1}(M \setminus M_0) \neq \emptyset$$

Indeed, if this intersection were empty, then $dist(M_0, exp^{-1}(M \setminus M_0)) > C$ and hence $T(M, \frac{C}{2})$ would be embedded. This is a contradiction by Corollary 1. In other words, $T(M,C)$ contains graphical pieces of M above the zero section M_0.

(iii) If furthermore one has

$$\lim_{n \mapsto \infty} \frac{V(R,C)}{R^{n+1}} = \infty$$

then for C arbitrarily small there exists graphical pieces of M above its zero section.

2. MINIMAL DISTANCE EQUATION

Let M and $\bar{M}$ be two minimal hypersurfaces in $\mathbb{R}^{n+1}$. Let q be a point of $\bar{M}$ and let ϕ be the distance from q to M taken along a minimizing geodesic joining q to M. Let p be the endpoint of this geodesic on M. Let U be a neighborhood of p in M; Let $X : U \mapsto \mathbb{R}^{n+1}$ be the position vector of M around p and $\bar{X} : U \mapsto \mathbb{R}^{n+1}$ be the position vector of $\bar{M}$ around q. Let N (resp. $\bar{N}$), be the unit normal vector field of M (resp. $\bar{M}$). $\bar{X}$ is related to X via the equation

$$\bar{X} = X + \phi N.$$

We will choose on U the principal curvatures coordinates at p; let $e_1, \cdots e_n$ a orthonormal basis at p tangent at p to the curvature lines of respective principal curvatures λ_i and extend them in a neighborhood by parallel transport. We then obtain the following expression of the Minimal equation

$$\bar{\Delta}\phi + \sum_{i=1}^{n} \frac{\lambda_i}{1 + \lambda_i \phi} a_i = 0$$

where $\bar{\Delta}$ is the Laplacian of the metric induced by the immersion $\bar{X}$ and where

$$a_i = 2 < \bar{\nabla}\phi, e_i > . < e_i^T, N > + < \bar{N}, e_i >^2 - 1$$

The minimal equation in the bounded geometry case .

Let T be a tube around M of small radius ϵ. The distance function ϕ of any piece $\bar{M}$ at a distance less than ϵ from M satisfies a uniform elliptic equation, whose coefficients are uniformly bounded. Indeed, as M and $\bar{M}$ don't intersect, and as the curvatures are uniformly bounded, it is not hard to see that for ϕ small, $\bar{M}$ is a graph over a piece of M, and the slope of $\bar{M}$ over M is uniformly bounded, and tends to zero as ϕ tends to zero (cf. [S], where those estimates are proved for $n = 2$). Hence the embeddedness hypothesis implies C^1-bounds on ϕ.

This allows us to deduce a useful differential inequality from the minimal equation

Corollary. *If $|A|\phi \leq \epsilon$ for ϵ small enough and ϕ a positive solution to the minimal surface equation, then*

$$\Delta\phi + (1 - \epsilon)|A|^2\phi \leq 0.$$

3. AN ESTIMATE FOR POSITIVE FUNCTIONS OF $\Delta\phi + \alpha|A|^2\phi \leq 0$.

We apply here an ingenious method of Fischer-Colbrie (see [F], [SZ], [SY]) that gives a geometrical setting to estimate Jacobi functions. We apply this method to the weakly stable case.

We first need to know the expression of the Ricci curvature under a conformal change of the metric; Let u be a positive solution of

$$\Delta u + Vu \leq 0$$

on a domain $M \subset \mathbb{R}^{n+1}$ of dimension n. Let $|A|$ the norm of the second fundamental form and suppose $V \geq \alpha |A|^2$, where α will be determined later. Change conformally the metric of M

$$d\bar{s}^2 = v^2 ds^2, \quad v := F(u)$$

and let γ be a geodesic of the new metric $d\bar{s}^2$ joining a fixed point p_o to point p. of M. If γ is distance minimizing then the Hessian of the energy function is positive at γ in the space of paths of M. Hence as the second variation of length is positive , for any normal deformation Y with compact support along γ

$$\int_\gamma \left(\frac{d|Y|}{d\bar{s}}\right)^2 d\bar{s} \geq \int_\gamma < \bar{R}(T,Y)T, Y >_- d\bar{s}$$

where $\bar{T}$ is the unit tangent vector field along γ and where $\bar{R}$ is the curvature tensor of $\bar{M}$.

Let $\{\bar{e}_1 := T, \bar{e}_2, \cdots \bar{e}_n\}$ be an orthonormal basis of $T_{p_0}M$ for the new metric that is extended by parallel transport along γ. Let ϕ be a smooth function on γ with compact support and consider the normal deformation $Y := \phi \bar{e}_i$ to γ for any $i = 2 \cdots n$. Former inequality now becomes

$$\int_\gamma \left(\frac{d\phi}{d\bar{s}}\right)^2 d\bar{s} \geq \int_\gamma \phi^2 < \bar{R}(T, \bar{e}_i)\bar{T}, \bar{e}_i >_- d\bar{s}$$

Summing for all $i = 2, \ldots, n$

$$(n-1)\int_\gamma \left(\frac{d\phi}{d\bar{s}}\right)^2 d\bar{s} \geq \int_\gamma \phi^2 \bar{R}_{11} d\bar{s}$$

where $\bar{R}_{11}$ is the Ricci curvature in the new metric i.e.

$$\bar{R}_{11} := \sum_2^n < \bar{R}(\bar{T}, \bar{e}_i)\bar{T}, \bar{e}_i >_- .$$

We then express $\bar{R}$ in terms of u and R. we obtain

Proposition 2. *Let u be a positive solution of $\Delta u + Vu \leq 0$, $V = \alpha|A|^2$ for α a positive number greater than $1 - \epsilon$, on a complete submanifold $M^n \subset \mathbb{R}^N, n \leq 5$. Then for any p, $p' \in M$ there is a path γ joining p to p' such that such that*

$$c(n) \int_\gamma {\phi'}^2 ds \geq \int_\gamma \phi^2 (v^2 + q) ds.$$

where $v := \frac{d\ln(u)}{ds}$.

We deduce directly from proposition 2 the following proposition

Proposition 3. *Let u be a positive function defined on a curve γ of $M^n \subset \mathbb{R}^{n+1}, n \leq 4$ joining point o to p such that $\int (\phi')^2 ds \geq c \int \phi^2 |(\ln u)_s|^2 ds$; then there are constants $C(n)$ independent of γ and such that*

$$u(p) \geq \frac{u(o)}{r^{C(n)}}$$

where r represents the arc length of the curve γ in the metric of M. Moreover $C(n) \leq 2\sqrt{(e-1)c(n)}$.

In particular, if there is a $p \in M$ and an extrinsic ball around p, $B(p)$, such that $B(p) \cap M$ consists of a finite number of connected components then M is proper.

Indeed, let N be a graphical piece of hypersurface that lies in a tube around M in $\mathbb{R}^{n+1}, n \leq 4$ of radius c . N is determined by a positive height function u defined on a piece of M that equals c on the boundary of this piece and that satisfies in its interior $\Delta u + (1-\epsilon)|A|^2)u \leq 0$ (see paragraph 2). Proposition 3.2 gives a lower bound for u in terms of a distance to the boundary: fix a point p of the zero section M_0 of the normal bundle. Suppose that there exists at this point a small extrinsic ball around p $B(p, R_0)$ that contains only one component. Consider then the tube of constant radius ϵ: $T(R,\epsilon)$ for $R \geq R_0$. If $T(R,\epsilon) \setminus M_0$ does not contain pieces of M for any R then the tube is embedded. If not Let R_1 such that $T(R_1,\epsilon) \setminus M_0$ contains connected pieces of M : $M_{11}, \cdots, M_{1k_1}$ These pieces are graphs of functions u_i defined on disjoint subdomains of M_0 : $D_{11}, \ldots, D_{1k}$. The tubes $T(D_{1i}, u_{1i}), i = 1, \ldots, k$ are disjoint and embedded : if there were a piece N_i of M that lies between M_i and D_i,then by the embeddedness of M, N_i would cut $T(R_1,\epsilon) \setminus M_0$, hence N must be M_{1i}. We then consider an increasing sequence $\{R_j\}$ going to infinity and we apply the same reasoning to each $T(R_j,\epsilon)$. Finally the following tube around M:

$$\left[T(\epsilon) \setminus \left(\bigcup_{ji_j} T(D_{ji_j},\epsilon)\right)\right] \cup \left[\bigcup_{ji_j} T(D_{ji_j}, u_{ji_j})\right].$$

is embedded; hence M is proper.

References

[BGG] E. Bombieri, E de Giorgi and E Giusti, *Minimal cones and the Bernstein problem*, Inv. Math. **82** (1968), 243-269.

[CP] M. Do Carmo & C.K. Peng, *Stable complete minimal hypersurfaces*, Proc. of the 1980 Symposium of Differential Geometry, Beijing **1,2,3** (1982), 1349-1358.

[C] Q Chen, *Curvature estimates for stable minimal hypersurfaces in* $\mathbf{R}^4$ *and* $\mathbf{R}^5$, Ann. Global Anal. Geom. **19** (2001), 177-184.

[CSZ] H. Cao, Y. Shen & S. Zhu, *The structure of stable minimal hypersurfaces in* R^{n+1}, Math. Res. Lett Math. **4** (1997), 637-644.

[D] Do Carmo, *Riemannian Geometry*, Birkauser (1992).

[DP] M. do Carmo & C. Peng, *Stable complete minimal surfaces in* R^3 *are planes*, Bull. Amer. Math. Soc. (N.S.) 1 **6** (1979), 903-906.

[F] D. Fischer-Colbrie, *On Complete Minimal Surfaces with finite Morse Index in three Manifolds*, Inv. Math. **82** (1985), 121-132.

[G] A. Gray, Addison-Wesley (1990).

[LW] P. Li & J. Wang, *Minimal hypersurfaces with finite index*, Math. Res. Lett. **9** (2002), 95-103.

[LT] M. Lin & G Trudinger, *On Some Inequalities for Elementary Symmetric Functions*, Bull. Austral. Math. Soc. **50** (1994), 317-326.

[SZ] Y. Shen & S. Zhu, *Rigidity of stable minimal hypersurfaces*, Math. Annalen **309** (1997), 107-116.

[S] M. Soret, *minimal surfaces with bounded curvature in Euclidean space*, Communications in Analysis and Geometry **9** (2001), 921-950.

[SS] R. Schoen & L. Simon, *Seminar on Minimal Submanifolds*, Princeton University Press, Princeton, 1983.

[SSY] R. Schoen & L. Simon & S. T. Yau, *Curvature Estimates for Minimal Hypersurfaces*, Acta Math. **134** (1975), 275-288.

[SY] Y. Shen & R. Ye, *On Stable Minimal Surfaces in Manifolds of positive Bi-Ricci curvatures*, Duke Math. Jour. **85** (1996), 109-116.

Universita di L'Aquila, dipartimento di Matematica, L'Aquila ITALY & Universite F. Rabelais, dep. de Mathematiques Tours FRANCE

E-mail address: nelli@univaq.it& marc@gargan.math.univ-tours.fr