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RULED SURFACES AND THEIR GAUSS MAPS IN LORENTZ-MINKOWSKI SPACES

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Abstract. We examine the ruled surfaces in Lorentz-Minkowski spaces with finite type Gauss map or with pointwise 1-type Gauss map.

1. Introduction

Since late 1970's when B.-Y.Chen introduced the theory of finite type immersion, its study has been extended to the submanifolds of pseudo-Euclidean spaces, namely a pseudo-Riemannian submanifold M of an m-dimension pseudo-Euclidean space $\mathbb{E}^m_s$ with signature $(s, m-s)$ is said to be of *finite type* if its position vector field x can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is,

$$x = x_0 + x_1 + x_2 + \cdots + x_k, \tag{1.1}$$

where x_0 is a constant map, $x_1, \cdots, x_k$ non-constant maps such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in R$, $i = 1, 2, \cdots, k$ ([10]). If $\lambda_1, \lambda_2, \cdots, \lambda_k$ are different, then M is said to be of k-type. Similarly, we can apply this notion to a smooth map, for example the Gauss map G that is one of the most natural smooth maps on an n-dimensional pseudo-Riemannian submanifold M of $\mathbb{E}^m_s$. Thus, the Gauss map G is said to be of *finite type* if G is a finite sum of $\mathbb{E}^m_s$-valued eigenfunction of Δ ([10]). We also similarly define the notion of k-type Gauss map on M as usual.

In [9] B.-Y. Chen and P. Piccinni gave a general study of compact submanifolds with finite type Gauss map and classified compact surfaces of 1-type Gauss map in Euclidean spaces. F. Dillen, J. Pas and L. Verstraelen([15]) and C. Baikoussis and D. E. Blair ([3]) studied respectively surfaces of revolution and ruled surfaces in Euclidean 3- space with finite type Gauss map. C. Baikoussis, B.-Y. Chen and L. Verstraelen ([6]) showed that the only ruled surfaces with finite type Gauss map in an m-dimensional Euclidean space are cylinders over curves of finite type and planes.
Recently, D. W. Yoon([32]) studied rotation surfaces in the 4-dimensional Euclidean space with finite type Gauss map and obtain the complete classification theorem for those. Also, Y. H. Kim and D. W. Yoon([21]) investigated ruled surfaces with non-null base of finite type Gauss map in an m-dimensional Minkowski space $\mathbb{E}^m_1$.

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On the other hand, F. Dillen, J. Pas and L. Verstraelen([15]) proved that among the surface of revolution in $\mathbb{E}^3$, the only ones whose Gauss map satisfies the condition

$$\Delta G = AG \quad (A \in R^{3\times 3}) \tag{1.2}$$

are the planes, the sphere and the circular cylinders, where Δ is the Laplacian of the surface with respect to the induced metric and A is a fixed endomorphism of the ambient space. Also, C. Baikoussis and D. E. Blair([3]) studied the surfaces of revolution in $\mathbb{E}^3$ which satisfy the condition (1.2). D. W. Yoon([31]) recently studied translation surfaces in the 3-dimensional Minkowski space whose Gauss map G satisfies the condition (1.2).

For the Lorentz version S. M. Choi ([14]) showed that the only ruled surfaces with non-null base curve in a 3-dimensional Minkowski space $\mathbb{E}^3_1$ satisfying the condition (1.2) are locally the Euclidean plane, the Minkowski plane, the Lorentz hyperbolic cylinder, the Lorentz circular cylinder and the hyperbolic cylinder.

On the other hand, as an extension of 1-type Gauss map we can consider G is of pointwise 1-type, that is, $\Delta G = fG$, for some smooth function f. M. Choi and the author ([12]) introduced the notion of Gauss map of pointwise 1-type on ruled surfaces in the Euclidean 3-space. Also, the author and D. W. Yoon([20]) studied ruled surfaces in a 3-dimensional Minkowski space with pointwise 1-type Gauss map and obtained the complete classification theorems for those.

In this paper we will survey the result of ruled surfaces in Minkowski spaces satisfying $\Delta G = AG$ and finite type Gauss map so far.

2. Preliminaries

Let $\mathbb{E}^m_s$ be an m-dimensional pseudo-Euclidean space with signature $(s, m-s)$. Then the metric tensor $\tilde{g}$ in $\mathbb{E}^m_s$ has the form

$$\widetilde{g} = -\sum_{i=1}^{s} dx_i^2 + \sum_{i=s+1}^{m} dx_i^2 \tag{2.1}$$

where $(x_1, x_2, \cdots, x_m)$ is the standard rectangular coordinate system in $\mathbb{E}^m_s$. In particular, for $m \geq 2$, $\mathbb{E}^m_1$ is called the $Lorentz - Minkowski\ m - space$ or simply $Minkowski\ m - space$.

Let $x : M \longrightarrow \mathbb{E}^m_s$ be an isometric immersion of an oriented n-dimensional pseudo-Riemannian submanifold M into $\mathbb{E}^m_s$. Let $\tilde{\nabla}$ be the Levi-Civita connection of $\mathbb{E}^m_s$ and ∇ the induced connection on M. Then, the Gauss formula is obtained by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.2}$$

for vector fields X, Y tangent to M, where h is the second fundamental form of M in $\mathbb{E}^m_s$. The Weingarten formula is given by

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.3}$$

where A_ξ is the shape operator associated with the normal vector field ξ and D is the connection defined on the normal bundle.

Let $e_1, e_2, \cdots, e_m$ be an adapted local orthogonal frame in $\mathbb{E}^m_s$ such that $e_1, e_2, \cdots, e_n$ are tangent to M and $e_{n+1}, e_{n+2}, \cdots, e_m$ normal to M.

Let us now define the Gauss map G of a submanifold M into $G(n, m)$ in $\wedge^n \mathbb{E}^m_s$, where $G(n, m)$ is the Grassmannian manifold consisting of all oriented n-planes passing through the origin of $\mathbb{E}^m_s$ and $\wedge^n\mathbb{E}^m_s$ is the vector space obtained by the exterior product of n vectors in $\mathbb{E}^m_s$. In a natural way, for some positive integer k we can identify $\wedge^n\mathbb{E}^m_s$ with some pseudo-Euclidean space $\mathbb{E}^N_k$, where $N = \binom{m}{n}$. The map $G : M \longrightarrow G(n,m) \subset \mathbb{E}^N_k$ defined by $G(p) = (e_1 \wedge e_2 \wedge \cdots \wedge e_n)(p)$ is called the *Gauss map* of M that is a smooth map which carries a point p in M into the oriented n-plane in $\mathbb{E}^m_s$ obtained for the parallel translation of the tangent space of M at p in $\mathbb{E}^m_s$.

3. Submanifolds in Euclidean space with finite type Gauss map

In this section, we intend to investigate submanifolds of Euclidean space with finite type Gauss map. The class of submanifolds with finite type Gauss map is very large. For example, ruled surfaces of a Euclidean space, spheres, circular cylinders, rotation surfaces have finite type Gauss map. In particular, B.-Y. Chen and P. Piccinni studied submanifold of $\mathbb{E}^m$ with 1-type Gauss map.

Theorem 3.1([9]). *Let $x : M \longrightarrow \mathbb{E}^m$ be an isometric immersion of a compact, oriented Riemannian manifold M into $\mathbb{E}^m$. Then the Gauss map G is of 1-type if and only if M has constant scalar curvature, flat normal connection and parallel mean curvature vector in $\mathbb{E}^m$.*

If M is a hypersurface of $\mathbb{E}^m$, we have

Theorem 3.2([9]). *A compact hypersurface M of $\mathbb{E}^m$ has 1-type Gauss map G if and only if M is a hypersphere in E^m.*

Theorem 3.3([9]). *Let M be a compact surface in E^m. Then M has 1-type Gauss map if and only if M is one of the following surfaces :*
(a)*A sphere $S^2(r) \subset \mathbb{E}^3 \subset \mathbb{E}^4$; or*
(b) *The product of two plane circles $S^1(a) \times S^1(b) \subset \mathbb{E}^4 \subset \mathbb{E}^m$.*

An n-dimensional submanifold M of Euclidean space $\mathbb{E}^m$ is ruled if M is foliated by $(n-1)$-dimensional totally geodesic subspaces of $\mathbb{E}^m$. A complete classification of minimal ruled submanifolds of Eucliedan space has been obtained independently by U. Lumiste in [25], C. Thas in [28] and J. M. Barbosa, A. M. Dajczer and L.P. Jorge in [7]. They showed that a minimal ruled submanifold of a Euclidean space is a generalized helicoid. In [8] it is proved that a ruled surface of finite type in a Euclidean space is either a part of a cylinder over a curve of finite type or a heliold in E^m. In [16] F. Dillen extended the above result and proved ruled submanifold of finite type in Euclidean space is a cylinder on a curve of finite type or a generalized helicoid.

For finite type Gauss map, C. Baikoussis, B.-Y. Chen and L. Verstraelen([6]) proved the following :

Theorem 3.4([6]). *Cylinder over curves of finite type and planes are the only ruled surfaces in Euclidean m - space ($m \geq 3$) with finite type Gauss map.*

Let M be a surface of 3-dimension Euclidean space $\mathbb{E}^3$, then Gauss map G is a map $G : M \longrightarrow S^2(1)$, where $S^2(1)$ is the unit sphere in $\mathbb{E}^3$.

Consider the surface in Euclidean space satisfying the condition

$$\Delta G = AG \quad A \in Mat(R^{3\times 3}) \tag{3.2}$$

where $Mat(R^{3\times 3})$ is the set of 3×3-real matrices.

C. Baikoussis and D. E. Blair([3]) studied ruled surfaces and got the following result:

Theorem 3.5([3]). *Among the ruled surfaces in E^3, the only ones whose Gauss map satisfies the condition (3.2) are the planes and the circular cylinders.*

It is well-known that all surfaces given in the above satisfy the condition $\Delta G = \lambda G, \quad \lambda \in R$.

However, there may be some other surfaces satisfying $\Delta G = fG$, for some smooth function f. To avoid trivialities, we assume that $G \neq hc$ with h a scalar real function and c a constant vector. In such cases, we must again reside in the usual 1-type notion. For example, the helicoid is, up to a rigid motion, parameterized by

$$x(s,\ t) = (t\cos s, t\sin s, bs),\ \ b \neq 0 \tag{3.3}$$

and the Gauss map G is given by

$$G = \frac{1}{\sqrt{t^2+b^2}}(-b\sin s, b\cos s, -t) \tag{3.4}$$

(see [27]). Therefore, we easily see that the Gauss map satisfies

$$\Delta G = \frac{2b^2}{(t^2+b^2)^2} G \tag{3.5}$$

which is not of 1-type in usual sense. Thus, the following question naturally arises : Besides the helicoid, which other surfaces in Euclidean 3-space E^3 satisfy

$$\Delta G = fG \tag{3.6}$$

for some real valued function f ?

A submanifold M in Euclidean m-space $\mathbb{E}^m$ is said to have *pointwise 1-type Gauss map* if it satisfies (3.6).

For submanifolds with pointwise 1-type Gauss map ; M. Choi and Y. H. Kim recently proved the following

Theorem 3.6([12]). *The ruled surfaces in $\mathbb{E}^3$ with pointwise 1-type Gauss map are an open portion of the plane, the circular cylinder and the helicoid.*

4. Gauss map of submanifolds in pseudo-Euclidean space

In [22], the author and D. W. Yoon investigated ruled surfaces in an m-dimensional Minkowski space $\mathbb{E}^m_1$ for Lorentz version. They defined a ruled surface M in $\mathbb{E}^m_1$: Let I and J be open intervals containing 0 in the real line $\mathbb{R}$. Let $\alpha = \alpha(s)$ be a curve on in $\mathbb{E}^m_1$ defined on J and $\beta = \beta(s)$ a transversal vector field along α. So we have the parametrization for M

$$x = x(s,t) = \alpha(s) + t\beta(s), \quad s \in J, \quad t \in I.$$

We call α the *base curve* and β the *director curve*. In particular, if β is constant, the ruled surface is said to be *cylindrical*, and if it is not so, it is called *non-cylindrical*. In this paper, we consider that the base curve α is space-like or time-like. In the case, the director curve β can be naturally chosen so that it is orthogonal to α. Furthermore, we have ruled surfaces of five different kinds according to the character of the base curve α and the director curve β as follows: If the base curve α is space-like or time-like, then the ruled surface M is said to be of type M_+ or type M_-, respectively. Also, the ruled surface of type M_+ can be divided into three types. In the case that β is space-like, it is said to be of type M_+^1 or M_+^2 if β' is non-null or null, respectively. When β is time-like, β' must be space-like according to the causal character. In this case, M said to be of type M_+^3. On the other hand, for the ruled surface of type M_-, it is also said to be of type M_-^1 or M_-^2 if β' is non-null or null, respectively. Note that in the case of type M_- the director curve β is always space-like. The ruled surface of type M_+^1 or M_+^2 (resp. M_+^3, M_-^1 or M_-^2) is clearly space-like (resp. time-like).

They proved the following :

Theorem 4.1([22]). *Cylinders over curves of finite type and some flat non-cylindrical ruled surfaces of type M_+^2 are the only ruled surface over non-null base curve with finite type Gauss map in $E_1^m (m \geq 4)$.*

Remark. There are infinitely many flat non-cylindrical M_+^2-type ruled surfaces in $\mathbb{E}_1^m$ $(m \geq 4)$ with finite type Gauss map.

Example. Let α be a space-like curve of the form $\alpha = \alpha(s) = (e^s, e^s, 0, s)$ and β a vector field along α such that $\beta = \beta(s) = (s, s, 1, 0)$ in $\mathbb{E}_1^4$. Consider a ruled surface M parametrized by $x(s,t) = \alpha(s) + t\beta(s)$ on $s \in I$ and $t \in J$ for some open intervals I and J. Then, M is a non-cylindrical ruled surface of type M_+^2 and the Gauss map G of M satisfies $\Delta^2 G + \Delta G = 0$, $\Delta G \neq 0$.

Kim-Kim-Yoon ([24]) introduced an extended B-scroll which is a generalization of B-scroll in 3-dimensional Minkowski space $\mathbb{E}_1^3$ and they obtained

Theorem 4.2([23]). *Let M be a null scroll in an m-dimensional Minkowski space $\mathbb{E}_1^m$. Then M has 1-type Gauss map if and only if M is an open portion of an extended B-scroll.*

Kim-Kim-Yoon defined the notion of generalized B-scroll which includes the extended B-scroll while they were studying the ruled surfaces in Minkowski space with null 2-type Gauss map ([24]). Regarding the characterizing the generalized B-scroll, Kim-Kim-Yoon ([23]) proved

Theorem 4.3 (Characterization) *Let M be a ruled surface with finite type Gauss map in $\mathbb{E}_1^m$ if and only if M is an open part of either cylinders over non-null base curve of finite type or a generalized B-scroll.*

Remark. The ruled surfaces with finite type Gauss map in $\mathbb{E}_1^m$ have at most null 2-type Gauss map, i.e., either 1-type or null 2-type.

Let M be a pseudo-Riemannian surface in Minkowski 3-space $\mathbb{E}^3_1$. The Gauss map G can be regarded as a map $G : M \longrightarrow Q^2(\varepsilon) \subset \mathbb{E}^3_1$ which sends each point of M to the unit normal vector to M at the point, where $\varepsilon(=\pm 1)$ denotes the sign of the vector field G and $Q^2(\varepsilon)$ is a 2-dimensional space form as follows :

$$Q^2(\varepsilon) = \begin{cases} \mathbb{S}^2_1(1) = \{X \in \mathbb{E}^3_1 \mid \langle X, X\rangle = 1\} & \text{if} \quad \varepsilon = 1, \\ \mathbb{H}^2(-1) = \{X \in \mathbb{E}^3_1 \mid \langle X, X\rangle = -1\} & \text{if} \quad \varepsilon = -1. \end{cases}$$

Here, $S^2_1(1)$ is called the de Sitter space, $H^2(-1)$ the hyperbolic space in $\mathbb{E}^3_1$.
In this section, we investigate the submanifolds in Minkowski space $\mathbb{E}^3_1$ satisfying the condition

$$\Delta G = AG, \ A \in \mathrm{Mat}(R^{3\times 3}), \tag{4.1}$$

where $\mathrm{Mat}(R^{3\times 3})$ is the set of 3×3-real matrices.

In [3] C. Baikoussis and D. E. Blair studied ruled surfaces in $\mathbb{E}^3$ whose Gauss map satisfies (4.1). S. M. Choi ([14]) investigated the Lorentz version of the above result and she essentially obtains the similar result. In [1] L. J. Alias. et al extended Choi's classification of ruled surfaces in $\mathbb{E}^3_1$ whose Gauss map G satisfies the condition (4.1).

Theorem 4.4([14]). *The ruled surfaces with non-null base curve in a 3-dimensional Minkowski space $\mathbb{E}^3_1$ whose Gauss map G satisfies* (4.1) *are locally R^2, R^2_1, $S^1_1 \times R^1$, $R^1_1 \times S^1$ and $H^1 \times R^1$.*

Theorem 4.5([1]). *B − scrolls over null curves in a 3-dimensional Minkowski space $\mathbb{E}^3_1$ are the only ruled surfaces with null rulings satisfying the equation* (4.1).

Combining Theorem 4.4 and Theorem 4.5, we obtain

Theorem 4.6([1]). *A ruled surface M in $\mathbb{E}^3_1$ satisfies the equation* (4.1) *if and only if M is one of the following surface :*
(1) $R^2, R^2_1, S^1_1 \times R^1$, $R^1_1 \times S^1$ *and* $H^1 \times R^1$
(2) *B-scroll over a null ruling.*

On the other hand, it is well-known that all surfaces given in the above satisfy the condition $\Delta G = \lambda G$ ($\lambda \in R$) for Minkowski space. Y. H. Kim and D. W. Yoon studied submanifolds with pointwise 1-type Gauss map, in particular, the examples satisfying $\Delta G = fG$ are given abundently large (See, examples in [20]).

Theorem 4.7([20]). *Let M be a space-like ruled surface in a 3-dimensional Minkowski space. Then the Gauss map is of pointwise 1-type if and only if M is an open part of the following surfaces:*

1. *a Euclidean plane,*
2. *the hyperbolic cylinder,*
3. *the helicoid of the 1st kind,*
4. *the helicoid of the 2nd kind,*
5. *the conjugate of Enneper's surface of 2nd kind.*

Theorem 4.8([20]). *Let M be a time-like ruled surface in a 3-dimensional Minkowski space. Then the Gauss map is of pointwise 1-type if and only if M is an open part of one of the following surfaces:*

1. *a Minkowski plane,*
2. *the Lorentz circular cylinder,*
3. *the circular cylinder of index 1,*
4. *the helicoid of the 1st kind,*
5. *the helicoid of the 2nd kind,*
6. *the helicoid of the 3rd kind,*
7. *the conjugate of Enneper's surface of 2nd kind,*
8. *a flat B-scroll if $B^{'}$ is light-like,*
9. *a non-flat B-scroll if $B^{'}$ is non-null.*

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