Trends in Mathematics
Information Center for Mathematical Sciences
Volume 5, Number 2, December, 2002, Pages 87–96

CERTAIN INEQUALITIES FOR SUBMANIFOLDS IN LOCALLY CONFORMAL ALMOST COSYMPLECTIC MANIFOLDS

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ABSTRACT. We present certain inequalities involving several intrinsic invariants namely Chen's δ-invariant, scalar curvature, Ricci curvature and k-Ricci curvature, and main extrinsic invariant namely squared mean curvature for submanifolds in a locally conformal almost cosymplectic manifold with pointwise constant φ-sectional curvature. Applications of these inequalities give rise to several inequalities for slant, invariant, anti-invariant and CR-submanifolds. The equality cases are also discussed.

1. INTRODUCTION

"To establish simple relationships between the main intrinsic invariants and the main extrinsic invariants of a submanifold" is one of the most fundamental problems in submanifold theory as recalled by B.-Y. Chen ([6]). The main intrinsic invariants include Chen's δ-invariant, scalar curvature, Ricci curvature and k-Ricci curvature. The main extrinsic invariants are squared mean curvature and shape operator. For more details we refer to [7].

On the other hand, the theory of almost contact metric structures occupies one of the leading places in researches of modern differential geometry. It is due to a number of its applications in mathematical physics, e.g. in classical mechanics ([1]) and in theory of geometrical quantization ([12]). Furthermore, we mark out the richness of the internal contents of the theory of almost contact metric structures as well as the close connection of this theory with other sections of geometry. There is an interesting class of almost contact metric manifolds which are locally conformal to almost cosymplectic manifolds ([11]). These manifolds are called locally conformal almost cosymplectic manifolds ([16]). For more details about locally conformal almost cosymplectic manifolds we refer to [8], [9], [10], [14], [17] etc.

We select the class of locally conformal almost cosymplectic manifolds of pointwise constant φ-sectional curvature. The submanifolds selected are CR, slant, invariant and anti-invariant submanifolds. Based on the work in [20] and [21], several inequalities involving intrinsic and extrinsic invariants are presented. Equality cases are also discussed. Section 2 contains a brief introduction to locally conformal almost cosymplectic manifolds, while in the section 3 some necessary details about

2000 *Mathematics Subject Classification.* 53C40 (53C15, 53C25).
Key words and phrases. locally conformal almost cosymplectic manifold, invariant submanifold, anti-invariant submanifold, slant submanifold, CR-submanifold, Chen's δ-invariant, Ricci curvature, k-Ricci curvature, squared mean curvature, relative null space, totally umbilical submanifold, minimal submanifold, totally geodesic submanifold.
Received October 1, 2002.

different kind of submanifolds are presented. In the section 4, we give inequalities with left hand side containing scalar curvature and right hand side containing squared mean curvature for slant, invariant, anti-invariant and *CR*-submanifolds. The equality cases hold if and only if these submanifolds are totally geodesic. In the section 5, we obtain an inequality involving Ricci curvature and squared mean curvature along with discussion of equality cases. As applications several inequalities for slant, invariant, anti-invariant and *CR*-submanifolds are given. In the section 6, we present relationships between the k-Ricci curvature and the squared mean curvature for slant, invariant, anti-invariant and *CR*-submanifolds. In the section 7, for submanifolds tangent to the structure vector field ξ, we give a basic inequality involving its sectional curvatures, scalar curvature and its squared mean curvature. Some applications including inequalities between Chen's δ-invariant and the squared mean curvature are presented. The equality cases are also discussed.

2. Locally conformal almost cosymplectic manifolds

Let $\tilde{M}$ be a $(2m+1)$-dimensional almost contact manifold ([4]) endowed with an almost contact structure (φ, ξ, η) consisting of a $(1,1)$ tensor field φ, a vector field ξ, and a 1-form η satisfying $\varphi^2 = -I + \eta \otimes \xi$ and (one of) $\eta(\xi) = 1$, $\varphi\xi = 0$, $\eta \circ \varphi = 0$. The almost contact structure induces a natural almost complex structure J on the product manifold $\tilde{M} \times \mathbb{R}$ defined by $J(X, \lambda d/dt) = (\varphi X - \lambda\xi, \eta(X)d/dt)$, where X is tangent to $\tilde{M}$, t the coordinate of $\mathbb{R}$ and λ a smooth function on $\tilde{M} \times \mathbb{R}$. The almost contact structure is said to be *normal* ([18]) if the almost complex structure J is integrable. Let $\langle \, , \rangle$ be a compatible Riemannian metric with (φ, ξ, η), i.e., $\langle X, Y \rangle = \langle \varphi X, \varphi Y \rangle + \eta(X)\eta(Y)$ or equivalently, $\Phi(X,Y) \equiv \langle X, \varphi Y \rangle = -\langle \varphi X, Y \rangle$ along with $\langle X, \xi \rangle = \eta(X)$ for all $X, Y \in T\tilde{M}$. Then, $(\varphi, \xi, \eta, \langle \, , \rangle)$ is an almost contact metric structure on $\tilde{M}$, and $\tilde{M}$ is an almost contact metric manifold.

If the fundamental 2-form Φ and the 1-form η are closed, then $\tilde{M}$ is said to be *almost cosymplectic manifold* ([11]). A normal almost cosymplectic manifold is *cosymplectic* ([4]). An almost contact metric structure is cosymplectic if and only if $\tilde{\nabla}\varphi = 0$, where $\tilde{\nabla}$ is the Levi-Civita connection of the Riemannian metric $\langle , \rangle$. An example of a manifold which has an almost cosymplectic structure which is not cosymplectic, can be found in [15]. A conformal change of an almost contact metric structure is defined by $\varphi^* = \varphi$, $\xi^* = e^{-\rho}\xi$, $\eta^* = e^{\rho}\eta$, $\langle , \rangle^* = e^{2\rho} \langle , \rangle$, where ρ is a differentiable function [22]. $\tilde{M}$ is said to be a *locally conformal almost cosymplectic manifold* if every point of $\tilde{M}$ has a neighborhood $\mathcal{U}$ such that $\left(\mathcal{U}, \varphi^*, \xi^*, \eta^*, \langle , \rangle^*\right)$ is almost cosymplectic for some function ρ on $\mathcal{U}$. Equivalently, $\tilde{M}$ is locally conformal almost cosymplectic manifold if there exists a 1-form ω such that $d\Phi = 2\omega \wedge \Phi$, $d\eta = \eta \wedge \omega$ ([16]).

A plane section ϱ in $T_p\tilde{M}$ of an almost contact metric manifold $\tilde{M}$ is called a *φ-section* if $\varrho \perp \xi$ and $\varphi(\varrho) = \varrho$. $\tilde{M}$ is of *pointwise constant φ-sectional curvature* if at each point $p \in \tilde{M}$, the sectional curvature $\tilde{K}(\varrho)$ does not depend on the choice of the φ-section ϱ of $T_p\tilde{M}$, and in this case for $p \in \tilde{M}$ and for any φ-section ϱ of $T_p\tilde{M}$, the function c defined by $c(p) = \tilde{K}(\varrho)$ is called the *φ-sectional curvature* of $\tilde{M}$. A locally conformal almost cosymplectic manifold $\tilde{M}$ of dimension ≥ 5 is of pointwise constant φ-sectional curvature if and only if its curvature tensor $\tilde{R}$ is of the form ([16])

$$\begin{aligned} \tilde{R}(X,Y)Z &= \frac{c-3f^2}{4}\{\langle Y,Z\rangle X - \langle X,Z\rangle Y\} \\ &+ \frac{c+f^2}{4}\{2\langle X,\varphi Y\rangle \varphi Z + \langle X,\varphi Z\rangle \varphi Y - \langle Y,\varphi Z\rangle \varphi X\} \\ &+ \left(\frac{c+f^2}{4}+f'\right)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ \langle X,Z\rangle \eta(Y)\xi - \langle Y,Z\rangle \eta(X)\xi\} \end{aligned} \tag{1}$$

for all $X,Y,Z \in T\tilde{M}$, where f is the function such that $\omega = f\eta$, $f' = \xi f$; and c is the pointwise φ-sectional curvature of $\tilde{M}$.

3. Submanifolds

Let M be an n-dimensional Riemannian manifold. The scalar curvature τ at p is given by $\tau = \sum_{i<j} K_{ij}$, where K_{ij} is the sectional curvature of the plane section spanned by e_i and e_j at $p \in M$ for any orthonormal basis $\{e_1,\ldots,e_n\}$ for T_pM. For each point $p \in M$, let

$$(\inf K)(p) = \inf\{K(\pi) : \text{plane sections } \pi \subset T_pM\}.$$

Then, Chen's invariant $\delta_M(p)$ is given by

$$\delta_M(p) = \tau(p) - (\inf K)(p), \tag{2}$$

where τ is the scalar curvature of M (see also [7]). Let $\mathcal{L}$ be a k-plane section of T_pM and X a unit vector in $\mathcal{L}$. We choose an orthonormal basis $\{e_1,...,e_k\}$ of $\mathcal{L}$ such that $e_1 = X$. The Ricci curvature $\mathrm{Ric}_{\mathcal{L}}$ of $\mathcal{L}$ at X is given by

$$\mathrm{Ric}_{\mathcal{L}}(X) = K_{12} + K_{13} + \cdots + K_{1k}, \tag{3}$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i, e_j. $\mathrm{Ric}_{\mathcal{L}}(X)$ is called a *k-Ricci curvature*. The scalar curvature τ of the k-plane section $\mathcal{L}$ is given by

$$\tau(\mathcal{L}) = \sum_{1\le i<j\le k} K_{ij}. \tag{4}$$

For each integer k, $2 \le k \le n$, the Riemannian invariant θ_k on an n-dimensional Riemannian manifold M is defined by

$$\theta_k(p) = \left(\frac{1}{k-1}\right)\inf_{\mathcal{L},X} \mathrm{Ric}_{\mathcal{L}}(X), \quad p \in M, \tag{5}$$

where $\mathcal{L}$ runs over all k-plane sections in T_pM and X runs over all unit vectors in $\mathcal{L}$.

Now, if M is immersed in an m-dimensional Riemannian manifold $\left(\tilde{M}, \langle\,,\rangle\right)$, then Gauss and Weingarten formulas are given respectively by $\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y)$ and $\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$ for all $X,Y \in TM$ and $N \in T^{\perp}M$, where $\tilde{\nabla}$, ∇ and $\nabla^{\perp}$ are Riemannian, induced Riemannian and induced normal connections in $\tilde{M}$, M and the normal bundle $T^{\perp}M$ of M respectively, and σ is the second fundamental form related to the shape operator A_N in the direction of N by $\langle\sigma(X,Y),N\rangle = \langle A_N X, Y\rangle$. The Gauss equation is

$$\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) - \langle\sigma(X,W),\sigma(Y,Z)\rangle + \langle\sigma(X,Z),\sigma(Y,W)\rangle \tag{6}$$

for all $X, Y, Z, W \in TM$, where $\tilde{R}$ and R are the curvature tensors of $\tilde{M}$ and M respectively. Let $\{e_1, ..., e_n\}$ be an orthonormal basis of the tangent space T_pM. The mean curvature vector H at $p \in M$ is expressed by $nH = \operatorname{trace}(\sigma)$. The submanifold M is *totally geodesic* in $\tilde{M}$ if $\sigma = 0$, and *minimal* if $H = 0$. If $\sigma(X, Y) = \langle X, Y\rangle H$ for all $X, Y \in TM$, then M is *totally umbilical*. We put

$$\sigma_{ij}^r = \langle \sigma(e_i, e_j), e_r\rangle \quad \text{and} \quad \|\sigma\|^2 = \sum_{i,j=1}^{n} \langle \sigma(e_i, e_j), \sigma(e_i, e_j)\rangle,$$

where e_r belongs to an orthonormal basis $\{e_{n+1}, ..., e_m\}$ of the normal space $T_p^\perp M$. At each point $p \in M$ the squared second fundamental form and the squared mean curvature satisfy

$$\begin{aligned} \|\sigma\|^2 &= \frac{1}{2}n^2\|H\|^2 + \frac{1}{2}\sum_{r=n+1}^{m}(\sigma_{11}^r - \sigma_{22}^r - \cdots - \sigma_{nn}^r)^2 \\ &\quad + 2\sum_{r=n+1}^{m}\sum_{j=2}^{n}(\sigma_{1j}^r)^2 - 2\sum_{r=n+1}^{m}\sum_{2\le i<j\le n}\left(\sigma_{ii}^r\sigma_{jj}^r - (\sigma_{ij}^r)^2\right). \end{aligned} \tag{7}$$

The *relative null space* of M at a point $p \in M$ is defined by ([6])

$$N_p = \{X \in T_pM |\, \sigma(X, Y) = 0 \text{ for all } Y \in T_pM\}.$$

Next, let M be an n-dimensional submanifold in an almost contact metric manifold. For a vector field X in M, we put

$$\varphi X = PX + FX, \qquad PX \in TM,\ FX \in T^\perp M.$$

Thus, P is an endomorphism of the tangent bundle of M and satisfies $\langle X, PY\rangle = -\langle PX, Y\rangle$ for all $X, Y \in TM$. The squared norm of P is defined by

$$\|P\|^2 = \sum_{i,j=1}^{n} \langle e_i, Pe_j\rangle^2$$

for any local orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ for T_pM. For a plane section $\varrho \subset T_pM$ at a point $p \in M$,

$$\alpha(\varrho) = \langle e_1, Pe_2\rangle^2 \qquad \text{and} \qquad \beta(\varrho) = (\eta(e_1))^2 + (\eta(e_2))^2$$

are real numbers in the closed unit interval $[0, 1]$, which are independent of the choice of the orthonormal basis $\{e_1, e_2\}$ of ϱ.

A submanifold M of an almost contact metric manifold with $\xi \in TM$ is called a *semi-invariant submanifold* ([3]) or a (*contact*) *CR-submanifold* ([23]) if there exists two differentiable distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ on M such that **(i)** $TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \mathcal{E}$, **(ii)** the distribution $\mathcal{D}$ is invariant by φ, i.e., $\varphi(\mathcal{D}) = \mathcal{D}$, and **(iii)** the distribution $\mathcal{D}^\perp$ is anti-invariant by φ, i.e., $\varphi(\mathcal{D}^\perp) \subseteq T^\perp M$.

A submanifold M tangent to ξ is said to be *invariant* or *anti-invariant* ([23]) according as $F = 0$ or $P = 0$. Thus, a *CR*-submanifold is invariant or anti-invariant according as $\mathcal{D}^\perp = \{0\}$ or $\mathcal{D} = \{0\}$. A proper *CR*-submanifold is neither invariant nor anti-invariant.

For each non zero vector $X \in T_pM$, such that X is not proportional to ξ_p, we denote the angle between φX and T_pM by $\theta(X)$. Then M is said to be *slant* ([5],[13]) if the angle $\theta(X)$ is constant, i.e., it is independent of the choice of $p \in M$ and $X \in T_pM - \{\xi\}$. The angle θ of a slant immersion is called the *slant angle* of the immersion. Invariant and anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \pi/2$ respectively. A *proper* slant immersion is neither invariant nor anti-invariant.

4. Scalar curvature

Let M be an n-dimensional submanifold in a locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature c such that the structure vector field ξ is tangent to M. In view of (1) and (6), the scalar curvature and the mean curvature of M satisfy ([20])

$$\begin{aligned} n^2 \|H\|^2 &= 2\tau + \|\sigma\|^2 - \frac{1}{4}n(n-1)\left(c - 3f^2\right) \\ & -\frac{3}{4}\|P\|^2\left(c+f^2\right) + 2(n-1)\left(\frac{c+f^2}{4} + f'\right). \end{aligned} \tag{8}$$

In an n-dimensioanl θ-slant submanifold M of an almost contact metric manifold, we get

$$\|P\|^2 = (n-1)\cos^2\theta. \tag{9}$$

If M is a *CR*-submanifold, then

$$\|P\|^2 = 2h = \dim(\mathcal{D}). \tag{10}$$

Thus, we are able to state the following

Theorem 4.1. [21]*For an n-dimensional submanifold M in a locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature c such that $\xi \in TM$, the following statements are true.*

1. *We have*

$$\begin{aligned} \tau &\leq \frac{1}{2}n^2\|H\|^2 + \frac{1}{8}n(n-1)\left(c-3f^2\right) \\ & +\frac{1}{8}\{3\|P\|^2 - 2(n-1)\}\left(c+f^2\right) - (n-1)f'. \end{aligned} \tag{11}$$

2. *If M is a θ-slant submanifold, then*

$$\tau \leq \frac{1}{2}n^2\|H\|^2 + \frac{n-1}{8}\{n(c-3f^2) + (3\cos^2\theta - 2)(c+f^2) - 8f'\}. \tag{12}$$

3. *If M is an invariant submanifold, then*

$$\tau \leq \frac{1}{2}n^2\|H\|^2 + \frac{n-1}{8}\{(n+1)c - (3n-1)f^2 - 8f'\}. \tag{13}$$

4. *If M is an anti-invariant submanifold, then*

$$\tau \leq \frac{1}{2}n^2\|H\|^2 + \frac{n-1}{8}\{(n-2)c - (3n+2)f^2 - 8f'\}. \tag{14}$$

5. *If M is a CR-submanifold, then*

$$\begin{aligned} \tau &\leq \frac{1}{2}n^2\|H\|^2 + \frac{1}{8}n(n-1)\left(c-3f^2\right) \\ & +\frac{1}{8}\{6h - 2(n-1)\}\left(c+f^2\right) - (n-1)f', \end{aligned} \tag{15}$$

where $2h = \dim(\mathcal{D})$.

6. *The equality cases of* (11), (12), (13), (14) *and* (15) *hold if and only if M is totally geodesic.*

5. Ricci curvature

In [6], B.-Y. Chen established a relationship between Ricci curvature and the squared mean curvature for a submanifold in a real space form. Here, we present similar results for several kind of submanifolds in a locally conformal almost cosymplectic manifold.

Theorem 5.1. [21] *Let M be an n-dimensional submanifold in a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature c tangential to the structure vector field ξ. Then, the following statements are true.*
(i) *For each unit vector $X \in T_pM$, we have*

$$\begin{aligned} Ric(X) \leq\ & \frac{1}{4}\left\{n^2\|H\|^2 + (n-1)\left(c-3f^2\right) \right. \\ & \left. + \left(3\|PX\|^2 - (n-2)\,\eta(X)^2 - 1\right)\left(c+f^2\right)\right\} \\ & - \left(1 + (n-2)\,\eta(X)^2\right) f'. \end{aligned} \tag{16}$$

(ii) *For $H(p)=0$, a unit vector $X \in T_pM$ satisfies the equality case of* (16) *if and only if X belongs to the relative null space N_p at p.*
(iii) *The equality case of* (16) *holds identically for all unit vectors $X \in T_pM$, if and only if either $n=2$ and p is a totally umbilical point or p is a totally geodesic point.*

The above theorem implies the following results for slant, invariant and anti-invariant submanifolds.

Theorem 5.2. [21] *Let M be an n-dimensional submanifold in a locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature c such that $\xi \in TM$. Then, the following statements are true.*
1. *If M is θ-slant, then for each unit vector $X \in T_pM$, we have*

$$\begin{aligned} Ric(X) \leq\ & \frac{1}{4}\left\{n^2\|H\|^2 + (n-1)\left(c-3f^2\right) \right. \\ & \left. + \left(3\cos^2\theta - \left(n+3\cos^2\theta-2\right)\eta(X)^2 - 1\right)\left(c+f^2\right)\right\} \\ & - \left(1 + (n-2)\,\eta(X)^2\right) f'. \end{aligned} \tag{17}$$

2. *If M is invariant, then for each unit vector $X \in T_pM$, we have*

$$\begin{aligned} Ric(X) \leq\ & \frac{1}{4}\{n^2\|H\|^2 + (n-1)(c-3f^2) + (2-(n+1)\eta(X)^2)(c+f^2)\} \\ & - \left(1 + (n-2)\,\eta(X)^2\right) f'. \end{aligned} \tag{18}$$

3. *If M is anti-invariant, then for each unit vector $X \in T_pM$, we have*

$$Ric(X) \leq \frac{1}{4}\{n^2\|H\|^2 + (n-1)(c-3f^2) - (1+(n-2)\eta(X)^2)(c+f^2+4f')\}. \tag{19}$$

4. *If $H(p)=0$, a unit vector $X \in T_pM$ satisfies the equality case of* (17), (18) *and* (19) *if and only if $X \in N_p$.*
5. *The equality case of* (17), (18) *and* (19) *holds identically for all unit vectors $X \in T_pM$, if and only if either $n=2$ and p is a totally umbilical point or p is a totally geodesic point.*

We also have the following

Corollary 5.3. [21] *Let M be an n-dimensional CR-submanifold in a locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature c. Then, the following statements are true.*
1. *For each unit vector $X \in \mathcal{D}$, we have*

$$4Ric(X) \leq n^2\|H\|^2 + (n+1)c - (3n-5)f^2 - 4f^{'}. \tag{20}$$

2. *For each unit vector $X \in \mathcal{D}^{\perp}$, we have*

$$4Ric(X) \leq n^2\|H\|^2 + (n-2)c - (3n-2)f^2 - 4f'. \tag{21}$$

3. *If $H(p) = 0$, a unit vector $X \in \mathcal{D}$ (resp. $\mathcal{D}^{\perp}$) satisfies the equality case of* (20) *(resp.* (21)*) if and only if $X \in N_p$.*

6. k-Ricci curvature

In this section, we give a relationship between the k-Ricci curvature and the squared mean curvature for submanifolds in a locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature c such that $\xi \in TM$.

Theorem 6.1. [21] *Let M be an n-dimensional submanifold in a $(2m+1)$-dimensioanl locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature c such that $\xi \in TM$. Then we have*

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - \frac{c-3f^2}{4} - \frac{3\|P\|^2(c+f^2)}{4n(n-1)} + \frac{2}{n}\left(\frac{c+f^2}{4} + f^{'}\right). \tag{22}$$

Next, we have the following

Theorem 6.2. [21] *Let M be an n-dimensional submanifold in a locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature c such that $\xi \in TM$. Then, for each integer k, $2 \leq k \leq n$, and every point $p \in M$, we have*

$$\|H\|^2 \geq \theta_k(p) - \frac{c-3f^2}{4} - \frac{3\|P\|^2(c+f^2)}{4n(n-1)} + \frac{2}{n}\left(\frac{c+f^2}{4} + f^{'}\right). \tag{23}$$

As an application, we have the following Corollary.

Corollary 6.3. [21] *Let M be an n-dimensional submanifold in a locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature c such that $\xi \in TM$, and let k be any integer such that $2 \leq k \leq n$. Then at each point $p \in M$, we have the following statements.*
1. *If M is θ-slant, then*

$$\|H\|^2 \geq \theta_k - \frac{c-3f^2}{4} - \frac{3(c+f^2)\cos^2\theta}{4n} + \frac{2}{n}\left(\frac{c+f^2}{4} + f^{'}\right). \tag{24}$$

2. *If M is invariant, then*

$$\|H\|^2 \geq \theta_k - \frac{c-3f^2}{4} - \frac{3(c+f^2)}{4n} + \frac{2}{n}\left(\frac{c+f^2}{4} + f^{'}\right). \tag{25}$$

3. *If M is anti-invariant, then*

$$\|H\|^2 \geq \theta_k - \frac{c-3f^2}{4} + \frac{2}{n}\left(\frac{c+f^2}{4}+f^{'}\right). \tag{26}$$

4. *If M is a CR-submanifold, then*

$$\|H\|^2 \geq \theta_k - \frac{c-3f^2}{4} - \frac{6h\left(c+f^2\right)}{4n\left(n-1\right)} + \frac{2}{n}\left(\frac{c+f^2}{4}+f^{'}\right), \tag{27}$$

where $2h = \dim(\mathcal{D})$.

7. Chen's δ-invariant

A basic inequality is presented in the following

Theorem 7.1. [20] *Let M be an n-dimensional $(n \geq 3)$ submanifold isometrically immersed in a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature c such that the structure vector field ξ is tangent to M. Then, for each point $p \in M$ and each plane section $\varrho \subset T_pM$, we have*

$$\begin{aligned}\tau - K(\varrho) \quad \leq \quad & \frac{n^2(n-2)}{2n}\|H\|^2 - \frac{1}{2}(n+1)(n-2)f^2 - (n-1-\beta(\varrho))f' \\ & + \frac{\left(c+f^2\right)}{8}\Big(3\|P\|^2 - 6\alpha(\varrho) + 2\beta(\varrho) + n(n-3)\Big).\end{aligned} \tag{28}$$

The equality in (28) *holds at $p \in M$ if and only if there exists an orthonormal basis $\{e_1,\dots,e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1},\dots,e_{2m+1}\}$ of $T_p^{\perp}M$ such that* **(a)** *$\varrho = Span\{e_1,e_2\}$ and* **(b)** *the forms of shape operators $A_r \equiv A_{e_r}$, $r = n+1,\dots,2m+1$, become*

$$A_{n+1} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & (\lambda+\mu) I_{n-2} \end{pmatrix}, \tag{29}$$

$$A_r = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 \\ h_{12}^r & -h_{11}^r & 0 \\ 0 & 0 & 0_{n-2} \end{pmatrix}, \qquad r = n+2,\dots,2m+1. \tag{30}$$

As applications we have the following results.

Theorem 7.2. [20] *Let M be an n-dimensional $(n > 2)$ submanifold isometrically immersed in a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature c such that the structure vector field ξ is tangent to M. If $c < -f^2$, then*

$$\begin{aligned}\delta_M \quad \leq \quad & \frac{n^2(n-2)}{2n}\|H\|^2 - \frac{1}{2}(n+1)(n-2)f^2 - (n-1)f' \\ & + \frac{1}{2}n(n-3)\frac{\left(c+f^2\right)}{4}.\end{aligned} \tag{31}$$

The equality in (31) *holds if and only if M is a semi-invariant submanifold with $rank(P) = 2$ or equivalently* $\dim(\mathcal{D}) = 2$.

Theorem 7.3. [20] *Let M be an n-dimensional $(n > 2)$ submanifold isometrically immersed in a $(2m + 1)$-dimensional locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature c such that $\xi \in TM$. If $c > -f^2$, then*

$$\begin{aligned}\delta_M \quad &\leq \quad \frac{n^2(n-2)}{2n}\|H\|^2 - \frac{1}{2}(n+1)(n-2)f^2 - (n-2)f' \\ &\quad + \frac{1}{2}(n-1)(n+1)\frac{(c+f^2)}{4}.\end{aligned} \tag{32}$$

The equality in (32) holds if and only if M is an invariant submanifold and $\beta = 1$.

Theorem 7.4. [20] *Let M be an n-dimensional $(n > 2)$ submanifold isometrically immersed in a $(2m + 1)$-dimensional normal locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature $c > -f^2$ such that $\xi \in TM$ and*

$$\delta_M = \frac{n^2(n-2)}{2n}\|H\|^2 - \frac{1}{2}(n+1)(n-2)f^2 - (n-2)f' + \frac{1}{2}(n-1)(n+1)\frac{(c+f^2)}{4}.$$

Then, M is a totally geodesic locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c.

Now for the case $c = -f^2$, we have the following pinching result.

Corollary 7.5. [20] *Let M be an n-dimensional $(n > 2)$ submanifold isometrically immersed in a $(2m + 1)$-dimensional locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature $c = -f^2$ such that $\xi \in TM$. Then, we have*

$$\begin{aligned}\delta_M \quad &\leq \quad \frac{n^2(n-2)}{2n}\|H\|^2 - \frac{1}{2}(n+1)(n-2)f^2 - (n-2)f', \qquad f' > 0, \\ \delta_M \quad &\leq \quad \frac{n^2(n-2)}{2n}\|H\|^2 - \frac{1}{2}(n+1)(n-2)f^2 - (n-1)f', \qquad f' < 0.\end{aligned}$$

Acknowledgement. The author is thankful to Professor Dr. Jeong-Hyeong Park, Honam University, Korea for inviting to deliver this work and for partial support by Brain Korea-21 Project at Chonnam National University, Korea.

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