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AN ESTIMATION FOR THE EIGENVALUES OF THE TRANSVERSAL DIRAC OPERATOR

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ABSTRACT. On a foliated Riemannian manifold $(M, g_M, \mathcal{F})$ with a transverse spin foliation $\mathcal{F}$, we estimate a lower bound for the square of the eigenvalues of the transversal Dirac operator D_{tr}.

1. Introduction

In 1963, A. Lichnerowicz([12]) proved that on a Riemannian spin manifold the square of the Dirac operator D is given by

$$D^2 = \Delta + \frac{\sigma}{4},$$

where Δ is the positive spinor Laplacian and σ the scalar curvature. In 1980, T. Friedrich([3]) gave a lower bound for the square of the eigenvalues of the Dirac operator in the above equation, as follows:

$$\lambda^2 \geq \frac{n}{4(n-1)}\sigma_0,$$

where $\sigma_0 = \min\sigma$. He also proved, in the limiting case, that the manifold is an Einstein.

In 1988, J. Brüning and F. W. Kamber ([2]) defined the transversal Dirac operator D_{tr} on M and proved the following equation:

$$D_{tr}^2 = \nabla_{tr}^*\nabla_{tr} + \mathcal{R}_\nabla + \mathcal{K}_\nabla,$$

where $\mathcal{R}_\nabla$ is an endomorphism containing the curvature data and K_∇ a function containing the mean curvature of the leaves.

This paper is a survay on the transversal Dirac operator D_{tr} and its eigenvalues on the foliated Riemannian manifold M, which is basied on the works [6,7,8] of the researcher.

2. Basic Laplacian

Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$.

Let $\Omega_B^r(\mathcal{F})$ be the space of all *basic r-forms*, i.e.,

$$\Omega_B^r(\mathcal{F}) = \{\phi \in \Omega^r(M) |\ i(X)\phi = 0,\ \theta(X)\phi = 0,\ \text{for } X \in \Gamma L\}.$$

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The foliation $\mathcal{F}$ is said to be *isoparametric* if $\kappa \in \Omega^1_B(\mathcal{F})$, where κ is a mean curvature form of $\mathcal{F}$(see [10, 14] for details). We already know that κ is closed, i.e., $d\kappa = 0$ if $\mathcal{F}$ is isoparametric ([14]). Since the exterior derivative preserves the basic forms (that is, $\theta(X)d\phi = 0$ and $i(X)d\phi = 0$ for $\phi \in \Omega^r_B(\mathcal{F})$), the restriction $d_B = d|_{\Omega^*_B(\mathcal{F})}$ is well defined. The *basic Laplacian* acting on $\Omega^*_B(\mathcal{F})$ is defined by

$$\Delta_B = d_B\delta_B + \delta_B d_B,$$

where δ_B is the adjoint operator of d_B. The differential operators d_B and δ_B are locally given by the following lemma. (See [1,8])

Lemma 2.1. *Let $\mathcal{F}$ be a Riemannian foliation. Then the operators d_B and δ_B on $\Omega^*_B(\mathcal{F})$ are given by*

$$\begin{aligned} d_B &= \sum_a \theta_a \wedge \nabla_{E_a}, \\ \delta_B &= -\sum_a i(E_a)\nabla_{E_a} + i(\kappa_B), \end{aligned}$$

where $\{E_a\}$ is a local orthonormal basic frame in Q and $\{\theta_a\}$ its g_Q-dual 1-form.

Lemma 2.1 can be proved by using the generalized Green's theorem ([15]). Also, Lemma 2.1 is proved in [1] by different method.

The basic cohomology $H^*_B(\mathcal{F}) = H(\Omega^*_B, d_B)$ plays the role of the De Rham cohomology of the leaf space $M/\mathcal{F}$ of the foliation.

Theorem 2.2. ([9]) *Let $\mathcal{F}$ be a transversally oriented Riemannian foliation on a compact orientable manifold (M, g_M). Assume g_M to be bundle-like with $\kappa \in \Omega^1_B(\mathcal{F})$. Then there is a decomposition into mutually orthogonal subspaces*

$$\Omega^r_B(\mathcal{F}) \equiv im d_B \oplus im\delta_B \oplus \mathcal{H}^r_B(\mathcal{F})$$

with finite dimensional $\mathcal{H}^r_B(\mathcal{F})$, where $\mathcal{H}^r_B(\mathcal{F}) = Ker\Delta_B$ is a set of the harmonic basic r-forms.

Also, the following vanishing theorem about the basic cohomology is well known.

Theorem 2.3. ([15]) *Let $\mathcal{F}$ be a Riemannian foliation of codimension $q \geq 2$ on a closed oriented Riemannian manifold (M, g_M) with bundle-like metric g_M. Then the following holds:*
(i) if $\rho^\nabla > 0$, then $H^1_B(\mathcal{F}) = 0$;
(ii) if $\mathcal{R}_\nabla > 0$, then $H^r_B(\mathcal{F}) = 0$ for $0 < r < q$.

If $\mathcal{F}$ is the foliation by points of M, the basic Laplacian is the ordinary Laplacian. Hence Theorem 2.2 and Theorem 2.3 are generalizations of the De-Rham Hodge Theory on the ordinary manifold.

3. The transverse Dirac operator

Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a transversally oriented Riemannian foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$. Let $SO(q) \to P \to M$ be the principal bundle of (oriented) transverse orthonormal framings. Then a *transverse spin structure* is a principal $Spin(q)$-bundle $\tilde{P}$ together with two sheeted covering $\xi : \tilde{P} \to P$ such that $\xi(p \cdot g) = \xi(p)\xi_0(g)$ for all $p \in \tilde{P},\ g \in Spin(q)$, where $\xi_0 : Spin(q) \to SO(q)$ is a covering. In this case, the foliation $\mathcal{F}$ is called a *transverse spin foliation*. We then define the vector bundle S associated with $\tilde{P}$ by

$$S = \tilde{P} \times_{Spin(q)} S_q, \tag{3.1}$$

where S_q is the irreducible spinor space associated to Q. The Hermitian metric on S is induced from g_Q, and the Riemannian connection ∇ on P can be lifted to one on $\tilde{P}$, in particular, to one on S, which will be denoted by the same letter. S is called the *foliated spinor bundle*. It is well known that the curvature transform R^S([11]) is given as

$$R^S_{XY}\Phi = \frac{1}{4}\sum_{a,b} g_Q(R^{\nabla}_{XY}E_a, E_b)E_a \cdot E_b \cdot \Phi \quad \text{for } X, Y \in TM. \tag{3.2}$$

On the foliated spinor bundle S, we have

$$\mathcal{R}_{\nabla} = \frac{1}{4}\sigma^{\nabla}, \tag{3.3}$$

$$\sum_a E_a \cdot R^S_{XE_a}\Phi = -\frac{1}{2}\rho^{\nabla}(X) \cdot \Phi \tag{3.4}$$

for $X \in \Gamma Q$([8]). The transverse Dirac operator D_{tr} on $S(\mathcal{F})$ is locally defined by

$$D_{tr}\Phi = \sum_a E_a \cdot \nabla_{E_a}\Phi - \frac{1}{2}\kappa \cdot \Phi, \tag{3.5}$$

where $\{E_a\}$ is an orthonormal basis of Q, the normal bundle of the foliation $\mathcal{F}$ and κ is a mean curvature form of $\mathcal{F}$(See [2,4,8]). From (3.5) we know ([4,8]) that on an isoparametric transverse spin foliation with $\delta\kappa = 0$, the transverse Dirac operator D_{tr} satisfies

$$D^2_{tr} = \nabla^*_{tr}\nabla_{tr} + \frac{1}{4}\sigma^{\nabla} + \frac{1}{4}|\kappa|^2. \tag{3.6}$$

If $\mathcal{F}$ is a minimal foliation, we know that

$$D^2_{tr} = d_B\delta_B + \delta_B d_B = \Delta_B. \tag{3.7}$$

4. The estimates

Let us introduce a new connection $\overset{f}{\nabla}$ on S as

$$\overset{f}{\nabla}_X \Phi = \nabla_X\Phi + f\pi(X) \cdot \Phi \quad \text{for } X \in TM, \tag{4.1}$$

where f is a real valued basic function on M and $\pi : TM \to Q$. Trivially, this connection $\overset{f}{\nabla}$ is a metric connection. Using this connection, we have

$$\overset{f}{\nabla}{}^*_{tr}\overset{f}{\nabla}_{tr} \Phi = \nabla^*_{tr}\nabla_{tr}\Phi - 2fD_{tr}\Phi - grad_{\nabla}(f) \cdot \Phi + qf^2\Phi, \tag{4.2}$$

where $grad_{\nabla}(f) = \sum_a E_a(f)E_a$ is a transversal gradient of f. From (3.6) and (4.2), we have

$$\overset{f}{\nabla}{}^*_{tr}\overset{f}{\nabla}_{tr} \Phi = D^2_{tr}\Phi - 2fD_{tr}\Phi - grad_{\nabla}(f) \cdot \Phi + (qf^2 - \frac{1}{4}(\sigma^{\nabla} + |\kappa|^2))\Phi. \tag{4.3}$$

Let $D_{tr}\Phi = \lambda\Phi(\Phi \neq 0)$. Note that for all $X \in \Gamma Q$ and $\Phi \in \Gamma S$,

$$< X \cdot \Phi, \Phi >= \overline{< \Phi, X \cdot \Phi >} = -\overline{< X \cdot \Phi, \Phi >}. \tag{4.4}$$

Hence (4.4) implies that $< grad_{\nabla}(f) \cdot \Phi, \Phi >$ is a pure imaginary. Hence we have

$$\| \overset{f}{\nabla}_{tr} \Phi\|^2 = \int_M (\lambda^2 - 2f\lambda + qf^2 - \frac{1}{4}(\sigma^{\nabla} + |\kappa|^2))|\Phi|^2, \tag{4.5}$$

$$< grad_{\nabla}(f) \cdot \Phi, \Phi >= 0. \tag{4.6}$$

If we put $f = \frac{\lambda}{q}$, then from (4.5), we have

$$\| \overset{f}{\nabla}_{tr} \Phi\|^2 = \int_M ((\frac{q-1}{q}\lambda^2 - \frac{1}{4}K_\sigma)|\Phi|^2, \tag{4.7}$$

where $K_\sigma = \sigma^{\nabla} + |\kappa|^2$. From (3.12), we have the following theorem (See [8]).

Theorem 4.1. *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with an isoparametric transverse spin foliation $\mathcal{F}$ of codimension $q > 1$ and bundle-like metric g_M with respect to $\mathcal{F}$. Assume that the mean curvature κ of $\mathcal{F}$ satisfies $\delta\kappa = 0$ and $\sigma^{\nabla} + |\kappa|^2 \geq 0$. Then the eigenvalue λ of the transverse Dirac operator D_{tr} satisfies*

$$\lambda^2 \geq \frac{1}{4}\frac{q}{q-1} min\{\sigma^{\nabla} + |\kappa|^2\}. \tag{4.8}$$

In the limiting case, $\mathcal{F}$ is minimal, transversally Einsteinian with constant scalar curvature.

Remark. If $\mathcal{F}$ is a point foliation, then the transversal Dirac operator is just a Dirac operator on an ordinary manifold. Therefore Theorem 4.1 is a generalization of the result on an ordinary manifold (cf.[3]).

By the conformal change of g_Q, we can obtain the shaper inequality than Theorem 4.1. (see [6] for details)

Theorem 4.2. *Under the same conditions as in Theorem 4.1 except for $q \geq 3$, we have*

$$\lambda^2 \geq \frac{1}{4}\frac{q}{q-1}(\mu_1 + min|\kappa|^2), \tag{4.9}$$

where μ_1 is the smallest eigenvalue of the basic Yamabe operator $Y_b = 4\frac{q-1}{q-2}\Delta_B + \sigma^{\nabla}$. In the limiting case, $\mathcal{F}$ is minimal, transversally Einsteinian with constant transversal scalar curvature.

On the other hand, on a Kähler spin foliation, the basic 2-form Ω play an improtant roles to estimate the eigenvalue of the transversal Dirac operator D_{tr}. In fact, Ω acts on S and then S splits into the orthogonal direct sum

$$S = S_0 \oplus S_1 \oplus \cdots \oplus S_n \quad (q = 2n).$$

By using this decompositon, we obtain the following theorem.(see [7])

Theorem 4.3. *Let $(M, g_M,)$ be a compact Riemannian manifold with Kähler spin foliation $\mathcal{F}$ of codimension $q = 2n$ and a bundle-like metric g_M with $\kappa \in \Omega_B^1$. Assume that the mean curvature form κ satisfies $\delta\kappa = 0$ and transversally holomorphic. If $\sigma^{\nabla} + |\kappa|^2 \geq 0$, then the eigenvalue λ of D_{tr} satisfies*

$$\lambda^2 \geq \frac{q+2}{4q} min(\sigma^{\nabla} + |\kappa|^2). \tag{4.10}$$

In the limiting case, $\mathcal{F}$ is minimal, transversally Einsteinian of odd complex codimension n with nonnegative constant transversal scalar curvature.

Problem. The estimates in Theorem 4.1, 4.2 and 4.3 imply that if the foliation $\mathcal{F}$ is not minimal, the inequality is strict. So, we can obtain a shaper estimates than the ones in theorems in case of non-minimal.

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