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HEAT CONTENT ASYMPTOTICS FOR SPECTRAL BOUNDARY CONDITIONS

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Abstract. We study the short time heat content asymptotics for spectral boundary conditions. The heat content coefficients are shown to be non-local and some preliminary results concerning the structure of the first few terms are given.

1. Introduction

The heat trace asymptotics for 'exotic' boundary conditions have recently attracted considerable interest; for a survey over this field see [12]. This article is devoted to the analogous questions for the heat content asymptotics. Whereas some formulas are available for the heat trace asymptotics with spectral boundary conditions [8, 10], nothing is known about the heat content asymptotics in this setting. We shall **postulate** in equation (2) the existence of an appropriate asymptotic series and then use special case calculations of spinors on the unit ball to study the heat content asymptotics defined by non-local spectral boundary conditions – these are preliminary results in an ongoing investigation.

Spectral boundary conditions were first introduced in the study of the index theorem for manifolds with boundary by Atiyah, Patodi, and Singer [1] who assumed that the structures involved were product near the boundary. Their work was later extended by Grubb and Seeley [15, 16, 17, 18, 19] to the general setting.

We briefly establish the notational conventions we shall employ and refer to [8, 10] for further details. Let $\mathcal{M}$ be a compact m-dimensional Riemannian manifold with smooth boundary $\partial\mathcal{M}$. We suppose given unitary vector bundles E_i over $\mathcal{M}$ and a first order elliptic complex:

$$P : C^\infty(E_1) \to C^\infty(E_2).$$

As such an elliptic complex need not admit local boundary conditions, it is natural to impose non-local spectral boundary conditions, which may be described as follows. Let γ be the leading symbol of P. Let ∇ be an auxiliary unitary connection with $\nabla\gamma = 0$; in many applications there is a natural choice available, but it is convenient to work quite generally for the moment. Let ∇_m denote covariant

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differentiation with respect to the inward geodesic unit normal. Near the boundary, we decompose

$$P = \gamma_m\{\nabla_m + B\},$$

where B is the *associated tangential first order operator* on $C^\infty(E_1|_{\partial\mathcal{M}})$. We also suppose given an auxiliary self-adjoint endomorphism Θ of $E_1|_{\partial\mathcal{M}}$ which we use to define a self-adjoint tangential operator A on $C^\infty(E_1|_{\partial\mathcal{M}})$ by setting:

$$A = \tfrac{1}{2}\{B + B^*\} + \Theta;$$

here the adjoint of B is taken with respect to the structures on the boundary. The endomorphism Θ arises naturally in the study of the signature and spin complexes where the metric is not product near the boundary and in this context is expressible in terms of the second fundamental form [9]; Θ compensates for non-canonical choice of ∇_m and plays an important role in the analysis in Section 3.

Let Π be orthogonal projection on the span of the eigenspaces corresponding to the non-negative eigenvalues of A. Let P_Π be the realization of P with the boundary condition Π. The index theorem for manifolds with boundary expresses $\text{index}(P_\Pi)$ in terms of characteristic forms integrated over $\mathcal{M}$, a compensating integral involving the second fundamental form over $\partial\mathcal{M}$, and a non-local term (the eta invariant) for the classic elliptic complexes [1, 9].

To simplify the discussion, we shall assume that $E_1 = E_2 = E$, that P is formally self-adjoint, and that P is of *Dirac type*; thus $D := P^2$ is formally self-adjoint and of *Laplace type*. The leading symbol γ of P is skew-adjoint and

$$\gamma(\xi)^2 = -|\xi|^2\text{Id} \qquad \text{for any} \qquad \xi \in T^*M.$$

$D_\mathcal{B}$ is the realization of D defined by the operator

$$\mathcal{B}\psi := \Pi\psi|_{\partial\mathcal{M}} \oplus \Pi P\psi|_{\partial\mathcal{M}}.$$

To avoid some technical fuss with the zero mode spectrum, we shall suppose that $\ker\{A\} = \{0\}$. We shall also assume that

$$\gamma_m A = -A\gamma_m \qquad \text{so} \qquad \gamma_m\Pi = \{\text{Id} - \Pi\}\gamma_m. \tag{1}$$

The operators P_Π and $D_\mathcal{B} = P_\Pi^2$ are then self-adjoint.

We now describe the fundamental solution of the heat equation. Let f_1 represent the initial "temperature" distribution of the manifold $\mathcal{M}$. The subsequent temperature distribution $h(x,t)$ for $t > 0$ is then given as the unique solution of the equations:

$$\partial_t h(x,t) = Dh(x,t), \quad \mathcal{B}h(x,t) = 0, \quad \text{and} \quad \lim_{t\to 0} h(x,t) = f_1(x).$$

Let $\langle\cdot,\cdot\rangle$ denote the Hermitian inner product on E. Let dx and dy be the Riemannian measures on $\mathcal{M}$ and $\partial\mathcal{M}$, respectively. Let f_2 represent the "specific heat" of the manifold. The total *heat energy content* of the manifold is given by:

$$\beta(f_1, f_2, D, \mathcal{B})(t) = \textstyle\int_\mathcal{M} dx\ \langle h(x,t), f_2(x)\rangle.$$

It is convenient at this point to review the situation for 'standard' boundary conditions. Let D be a formally self-adjoint second order operator of Laplace type whose realization is defined by Dirichlet or Robin boundary conditions $\mathcal{B}$. It is known that there is a complete short time asymptotic series of the form:

$$\beta(f_1, f_2, D, \mathcal{B}) \sim \sum_{n=0}^\infty \beta_n(f_1, f_2, D, \mathcal{B})t^{n/2}.$$

The coefficients β_n are called the *heat content asymptotics*.

Let Roman indices $\{i,j,k,l\}$ (resp. $\{a,b\}$) range from 1 to m (resp. $m-1$) and index an orthonormal frame for the tangent bundle $T\mathcal{M}$ (resp. $T\partial\mathcal{M}$). Let L_{ab} be the components of the second fundamental form and let R_{ijkl} be the components of the Riemann curvature tensor (with the sign convention $R_{1221}=+1$ on $S^2\subset\mathbb{R}^3$). Let ';' be multiple covariant differentiation with respect to the Levi-Civita connection on $\mathcal{M}$ and the natural connection defined by D, and let $\mathcal{E}$ be the endomorphism of E defined by D [3]. We adopt the Einstein convention and sum over repeated indices. We refer to [2, 3] for the proof of the following Lemma:

Lemma 1. *Let D be an operator of Laplace type on a compact Riemannian manifold $\mathcal{M}$ with smooth boundary $\partial\mathcal{M}$.*

(1) *If $\mathcal{B}\psi=\psi|_{\partial\mathcal{M}}$ defines Dirichlet boundary conditions, then*
   (a) $\beta_0(f_1,f_2,D,\mathcal{B})=\int_{\mathcal{M}}dx\ \langle f_1,f_2\rangle$.
   (b) $\beta_1(f_1,f_2,D,\mathcal{B})=-2\pi^{-1/2}\int_{\partial\mathcal{M}}dy\ \langle f_1,f_2\rangle$.
   (c) $\beta_2(f_1,f_2,D,\mathcal{B})=-\int_{\mathcal{M}}dx\ \langle Df_1,f_2\rangle$
   $+\int_{\partial\mathcal{M}}dy\ \{\langle\frac{1}{2}L_{aa}f_1,f_2\rangle-\langle f_1,f_{2;m}\rangle\}$
   (d) $\beta_3(f_1,f_2,D,\mathcal{B})=-2\pi^{-1/2}\int_{\mathcal{M}}dy\ \{\frac{2}{3}\langle f_{1;mm},f_2\rangle+\frac{2}{3}\langle f_1,f_{2;mm}\rangle$
   $-\langle f_{1;a},f_{2;a}\rangle+\langle\mathcal{E}f,f_2\rangle-\frac{2}{3}L_{aa}\langle f_{1;m},f_2\rangle-\frac{2}{3}L_{aa}\langle f_1,f_{2;m}\rangle$
   $+\langle(\frac{1}{12}L_{aa}L_{bb}-\frac{1}{6}L_{ab}L_{ab}+\frac{1}{6}R_{amam})f_1,f_2\rangle\}$.

(2) *If $\mathcal{B}\psi=(\nabla_m+S)\psi|_{\partial\mathcal{M}}$ defines Robin boundary conditions, then:*
   (a) $\beta_0(f_1,f_2,D,\mathcal{B})=\int_{\mathcal{M}}dx\ \langle f_1,f_2\rangle$.
   (b) $\beta_1(f_1,f_2,D,\mathcal{B})=0$.
   (c) $\beta_2(f_1,f_2,D,\mathcal{B})=-\int_{\mathcal{M}}dx\ \langle Df_1,f_2\rangle+\int_{\partial\mathcal{M}}dy\ \langle\mathcal{B}f_1,f_2\rangle$.
   (d) $\beta_3(f_1,f_2,D,\mathcal{B})=\frac{4}{3}\cdot\pi^{-1/2}\int_{\partial\mathcal{M}}dy\ \langle\mathcal{B}f_1,\mathcal{B}f_2\rangle$.

Lemma 1 shows that the heat content coefficients β_n for $n\le 3$ are locally computable for Dirichlet and Robin boundary conditions. In fact, all the coefficients β_n are locally computable for these boundary conditions. These invariants have been studied extensively [2, 3, 6, 20, 21]; we refer to [13] for a recent survey article.

We now return to the setting of spectral boundary conditions. We shall **assume** the existence of a similar asymptotic series for the realization of $D=P^2$ defined by spectral boundary conditions as $t\to 0$:

$$\beta(f_1,f_2,D,\mathcal{B})\sim\sum_{n=0}^{\infty}\beta_n(f_1,f_2,D,\mathcal{B})t^{n/2}. \tag{2}$$

A natural question then arises, why do we believe there are no log terms – after all, in the expansion of the heat trace $\mathrm{Tr}_{L^2}(e^{-tD_{\mathcal{B}}})$, log terms appear! Our answer is two-fold. First of all, the heat content asymptotics exhibit various structural simplifications compared to the heat trace asymptotics. One example is, as we shall see presently in Section 3, that in the special case of the Dirac operator on the unit ball the relevant universal constants do not depend on the dimension, as they did for the heat trace aymptotics. Another example is, that the heat content asymptotics do not show any signs of the loss of strong ellipticity for the case of oblique boundary conditions [11]. Furthermore, the heat trace asymptotics give rise to log terms above the dimension of the manifold in the series; for $n<m$, the asymptotic coefficients do not have log terms. So even if the situation is as for the heat trace, our results will still hold true for $n<m$. But this is certainly a question that merits further investigation.

The first coefficient β_0 is easily described. Since $\lim_{t\to 0} h(x,t) = f_1(x)$, we have $\lim_{t\to 0}\beta(f_1, f_2, D, \mathcal{B})(t) = \int_{\mathcal{M}} dx\ \langle f_1, f_2\rangle$ and thus, as for Dirichlet and Neumann boundary conditions,

$$\beta_0(f_1, f_2, D, \mathcal{B}) = \textstyle\int_{\mathcal{M}} dx\ \langle f_1, f_2\rangle. \tag{3}$$

Here is a brief outline to this paper. In Section 2, we discuss functorial properties of these invariants, show they are non-local, and outline what we believe the formula for β_1 and β_2 to be. These results are based on the special case computations in Section 3 giving a complete calculation of the heat content function for the Dirac operator on the unit ball with the standard metric.

2. Functorial Properties

The invariants β_n for Dirichlet and Robin boundary conditions have a number of functorial properties [2] which extend immediately to this setting:

Lemma 2. *Let $D_{\mathcal{B}} = P_\Pi^2$ be a self-adjoint operator of Laplace type defined by spectral boundary conditions as described above. Then:*

1. *We have $\beta_n(f_1, f_2, c^{-2}D, \mathcal{B}) = c^{m-n}\beta_n(f_1, f_2, D, \mathcal{B})$ for any $c \in \mathbb{R}^+$.*
2. *We have $\beta_n(f_1, f_2, D, \mathcal{B}) = \beta_n(f_2, f_1, D, \mathcal{B})$.*
3. *If $\mathcal{B}f_1 = 0$, then $\beta_n(f_1, f_2, D, \mathcal{B}) = -\frac{2}{n}\beta_{n-2}(Df_1, f_2, D, \mathcal{B})$.*

Proof. The operator $D_{\mathcal{B}}$ has a discrete spectral resolution $\mathcal{S}_{D,\mathcal{B}} := \{\psi_\nu, \lambda_\nu\}$ [15] with associated Fourier coefficients: $\sigma_\nu(\psi) := \int_{\mathcal{M}} dx\ \langle \psi, \psi_\nu\rangle$. We may then express:

$$\begin{aligned} h(x,t) &= \textstyle\sum_\nu e^{-t\lambda_\nu}\sigma_\nu(f_1)\psi_\nu(x) \qquad \text{so} \\ \beta(f_1, f_2, D, \mathcal{B})(t) &= \textstyle\sum_\nu e^{-t\lambda_\nu}\sigma_\nu(f_1)\sigma_\nu(f_2). \end{aligned} \tag{4}$$

The estimates of [15] show these series converge uniformly.

We take $\Theta(c) := c^{-1}\Theta$ to ensure the associated boundary condition is unchanged. Note that the Riemannian measure defined by the operator $c^{-2}D$ is $c^m dx$. Since $\mathcal{S}_{c^{-2}D,\mathcal{B}} = \{c^{-m/2}\psi_\mu, c^{-2}\lambda_\nu\}$ and $\sigma_\nu^c(f) = c^{m/2}\sigma_\nu(f)$, we have:

$$\beta(f_1, f_2, c^{-2}D, \mathcal{B})(t) = c^m\beta(f_1, f_2, D, \mathcal{B})(c^{-2}t).$$

Assertion (1) follows by equating powers of t in this equation. Since the roles of f_1 and f_2 are symmetric in display (4), assertion (2) follows. If $\mathcal{B}f_1 = 0$, then the boundary terms vanish and we can integrate by parts to compute:

$$\begin{aligned} \sigma_\nu(Df_1) &= \textstyle\int_{\mathcal{M}} dx\ \langle Df_1, \psi_\nu\rangle = \int_{\mathcal{M}} dx\ \langle f_1, D\psi_\nu\rangle = \lambda_\nu\sigma_\nu(f_1) \qquad \text{so} \\ -\partial_t\beta(f_1, f_2, D, \mathcal{B})(t) &= \textstyle\sum_\nu \lambda_\nu e^{-t\lambda_\nu}\sigma_\nu(f_1)\sigma_\nu(f_2) = \beta(Df_1, f_2, D, \mathcal{B})(t). \end{aligned}$$

We can now establish (3) by equating terms in the asymptotic expansions for $\partial_t\beta(f_1, f_2, D, \mathcal{B})$ and $\beta(Df_1, f_2, D, \mathcal{B})$. □

The following is an important observation.

Lemma 3. *The heat content coefficients for spectral boundary conditions are not locally computable*

Proof. If $\mathcal{B}f_1 = 0$ then $\beta_1(f_1, f_2, D, \mathcal{B}) = 0$ by Lemma 2. If β_1 is locally computable, then dimensional analysis (i.e. the scaling property given by assertion (1) of Lemma 2) implies that there must exist a universal constant $c_0(m)$ so that:

$$\beta_1(f_1, f_2, D, \mathcal{B}) = 2\pi^{-1/2}\textstyle\int_{\partial\mathcal{M}} dy\ c_0(m)\ \langle f_1, f_2\rangle.$$

Since generically there are eigensections with $\mathcal{B}f_1 = 0$ but $f_1|_{\partial\mathcal{M}} \neq 0$, we must have $c_0 = 0$; so far, the argument is exactly the same as for Robin boundary conditions given in [6] to prove Lemma 1 (2b). However, the calculation on the ball that we shall present in Section 3 shows the power $t^{1/2}$ is present in the asymptotic expansion with spectral boundary conditions. This contradiction establishes the Lemma. □

The coefficient β_0 is given by equation (3). Using the principle of *not feeling the boundary*, we see that the interior integrals defining β_n for spectral boundary conditions are the same as those defining β_n for either Dirichlet or Robin boundary conditions. Writing the interior term for β_2 in the form $\langle Df_1, f_2\rangle$ destroys the symmetry of Lemma 2 (2) so instead we use $\langle Pf_1, Pf_2\rangle$ and add suitable boundary correction terms. The boundary operator Π is a 0-th order operator; it is unaffected by rescaling. To ensure that properties (1) and (3) of Lemma 2 are satisfied, i.e.

$$\beta_n(f_1, f_2, c^{-2}D, \mathcal{B}) = c^{m-n}\beta_n(f_1, f_2, D, \mathcal{B}) \text{ and}$$
$$\beta_n(f_1, f_2, D, \mathcal{B}) = \beta_n(f_2, f_1, D, \mathcal{B})$$

we are lead to consider the following ansatz for β_1 and β_2 – the only non-local terms are introduced by the boundary condition.

Ansatz 4. *There exist universal constants $c_i = c_i(m)$ so that:*

(1) $\beta_1(f_1, f_2, D, \mathcal{B}) = 2\pi^{-1/2} \int_{\partial\mathcal{M}} dy\; c_0(m)\; \langle \Pi f_1, \Pi f_2\rangle$.
(2) $\beta_2(f_1, f_2, D, \mathcal{B}) = -\int_{\mathcal{M}} dx\; \langle Pf_1, Pf_2\rangle + \int_{\partial\mathcal{M}} dy\; \big\{c_1(m)\langle \Pi\gamma_m Pf_1, \Pi f_2\rangle$
$+c_1(m)\langle \Pi f_1, \Pi\gamma_m Pf_2\rangle + c_2(m)L_{aa}\langle \Pi f_1, \Pi f_2\rangle + c_3(m)\langle \Theta\Pi f_1, \Pi f_2\rangle\big\}$.

The remainder of this note is devoted to the evaluation of these coefficients:

Lemma 5. *We have $c_0(m) = -1$, $c_1(m) = 1$, $c_2(m) = \frac{1}{2}$, and $c_3(m) = 0$.*

It is interesting that these coefficients seem to be dimension free; the usual trick of dimension shifting employed in [3] does not work with spectral boundary conditions. By contrast, the coefficients in the heat trace asymptotics for spectral boundary conditions are highly dimension dependent [8, 10].

Proof. We must ensure the properties of Lemma 2 are satisfied. In particular $\beta_1(f_1, f_2, D, \mathcal{B})$ must vanish if Πf_1=0. The boundary term for β_1 must be homogeneous of degree 0. Since Π is a 0^{th} order operator, $\langle \Pi f_1, \Pi f_2\rangle$ has the correct homogeneity, is symmetric in the roles of $\{f_1, f_2\}$, and vanishes when $\mathcal{B}f_1 = 0$. This motivates the formula given in (1).

Consider $\beta_2(f_1, f_2, D, \mathcal{B})$. To preserve the interior symmetry, we use the interior integrand $-\langle Pf_1,\; Pf_2\rangle$ rather than $-\langle Df_1, f_2\rangle$. The corresponding integrals are related by the formula:

$$-\int_{\mathcal{M}} dx\langle Df_1, f_2\rangle \quad = \quad -\int_{\mathcal{M}} dx\langle Pf_1, Pf_2\rangle + \int_{\partial\mathcal{M}} dy\langle \gamma_m Pf_1, f_2\rangle.$$

If $\mathcal{B}f_1 = 0$, we know from Lemma 2 (3) that

$$\beta_2(f_1, f_2, D, \mathcal{B}) = -\beta_0(Df_1, f_2, D, \mathcal{B}).$$

Since $\Pi f_1 = \Pi Pf_1 = 0$ on $\partial\mathcal{M}$, we use equation (1) to see that

$$\gamma_m Pf_1 = \gamma_m(1 - \Pi)Pf_1 = \Pi\gamma_m Pf_1 \text{ on } \partial M.$$

Consequently, we find:

$$\begin{aligned} -\beta_0(Df_1,f_2,D,\mathcal{B}) &= -\int_{\mathcal{M}} dx\ \langle Pf_1,Pf_2\rangle + \int_{\partial\mathcal{M}} dy\ \langle \Pi\gamma_m Pf_1, f_2\rangle \\ &= -\int_{\mathcal{M}} dx\ \langle Pf_1,Pf_2\rangle + \int_{\partial\mathcal{M}} dy\ \langle \Pi\gamma_m Pf_1, \Pi f_2\rangle. \end{aligned} \tag{5}$$

This shows $c_1(m)=1$. Lemma 2 (2) shows we need to include $c_1(m)\langle\Pi f_1,\Pi\gamma_m Pf_2\rangle$ into Ansatz 4 (2). Lemma 2 (3) shows that apart from the invariants multiplied by $c_1(m)$ additional invariants must disappear if $\Pi f_1=\Pi Pf_1=0$. Applying Lemma 2 (1) this allows for the occurrence of the remaining terms in Ansatz 4 (2).

Replacing Θ by $\Theta+\varepsilon$ where ε is a small positive real constant does not change the spectral projection Π and hence does not change β_n. Thus $c_3(m)=0$. We postpone the evaluation of the remaining constants, $c_0(m)$ and $c_2(m)$, until Section 3. □

3. Calculations on the Ball

The eigenvalue problem on the ball is known [7, 10] and we will only summarize the relevant equations for the present context. We use the following representation of the γ-matrices projected along e_j:

$$\gamma_{a(m)} = \begin{pmatrix} 0 & \sqrt{-1}\cdot\gamma_{a(m-1)} \\ -\sqrt{-1}\cdot\gamma_{a(m-1)} & 0 \end{pmatrix} \text{ and}$$

$$\gamma_{m(m)} = \begin{pmatrix} 0 & \sqrt{-1}\cdot 1_{m-1} \\ \sqrt{-1}\cdot 1_{m-1} & 0 \end{pmatrix}.$$

We decompose $\nabla_j = e_j+\omega_j$ where $\omega_j=\frac14\Gamma_{jkl}\gamma_{k(m)}\gamma_{l(m)}$ is the connection 1 form of the spin connection – i.e.

$$\nabla_a = \frac1r\left(\begin{pmatrix} \tilde\nabla_a & 0 \\ 0 & \tilde\nabla_a\end{pmatrix} + \frac12\gamma_{m(m)}^{-1}\gamma_{a(m)}\right).$$

Let P and $\bar P$ be the Dirac operator on the ball and the sphere respectively. In the notation established above, the Dirac operator on the ball is

$$P = \left(\frac{\partial}{\partial x_m} - \frac{m-1}{2r}\right)\gamma_{m(m)} + \frac1r\begin{pmatrix} 0 & \sqrt{-1}\bar P \\ -\sqrt{-1}\bar P & 0\end{pmatrix}.$$

Let $\varphi_\pm$ and $\mathcal{Z}_\pm^{(n)}$ denote the eigen functions of P and $\bar P$ respectively,

$$P\varphi_\pm = \pm\mu\varphi_\pm,\ \ \bar P\mathcal{Z}_\pm^{(n)} = \pm\left(n+\frac{m-1}{2}\right)\mathcal{Z}_\pm^{(n)} \text{ for } n=0,1,2,...$$

A complete set of eigen functions is

$$\begin{aligned} \varphi_\pm^{(+)} &= \frac{C}{r^{(m-2)/2}}\begin{pmatrix} iJ_{n+m/2}(\mu r)\,Z_+^{(n)}(\Omega) \\ \pm J_{n+m/2-1}(\mu r)\,Z_+^{(n)}(\Omega)\end{pmatrix}, \text{ and} \\ \varphi_\pm^{(-)} &= \frac{C}{r^{(m-2)/2}}\begin{pmatrix} \pm J_{n+m/2-1}(\mu r)\,Z_-^{(n)}(\Omega) \\ iJ_{n+m/2}(\mu r)\,Z_-^{(n)}(\Omega)\end{pmatrix}, \text{where} \\ C &= J_{n+m/2}(\mu)^{-1} \end{aligned}$$

is the radial normalization constant. With the choice $\Theta=(m-1)/2\ 1_m$, the boundary operator A used to define spectral boundary conditions then reads

$$A = \begin{pmatrix} -\bar P & 0 \\ 0 & \bar P\end{pmatrix}.$$

It is easy to determine the discrete spectral resolution of A. One can show

$$
\begin{aligned}
A\begin{pmatrix} \mathcal{Z}_+^{(n)}(\Omega) \\ \mathcal{Z}_-^{(n)}(\Omega) \end{pmatrix} &= -\left(n+\frac{m-1}{2}\right)\begin{pmatrix} \mathcal{Z}_+^{(n)}(\Omega) \\ \mathcal{Z}_-^{(n)}(\Omega) \end{pmatrix} \text{ and} \\
A\begin{pmatrix} \mathcal{Z}_-^{(n)}(\Omega) \\ \mathcal{Z}_+^{(n)}(\Omega) \end{pmatrix} &= \left(n+\frac{m-1}{2}\right)\begin{pmatrix} \mathcal{Z}_-^{(n)}(\Omega) \\ \mathcal{Z}_+^{(n)}(\Omega) \end{pmatrix} \text{ for } n=0,1,....
\end{aligned}
$$

Thus there is no zero mode spectrum. Spectral boundary conditions suppress the non-negative spectrum of A, which yields the implicit eigenvalue equation

$$J_{n+m/2-1}(\mu)=0, \quad n=0,1,2,...$$

We will analyze the heat content asymptotics by considering the associated zeta function. Denoting by ϕ_k the full set of eigen functions, $\phi_k=(\varphi_\pm^{(+)},\varphi_\pm^{(-)})$, we write

$$\zeta(s,f_1,f_2,D,\mathcal{B})=\sum_k \lambda_k^{-s}\ (f_1,\phi_k)_{L^2}\ (\phi_k,f_2)_{L^2}.$$

The asymptotic coefficients β_n given in equation (2) are then given by:

$$\beta_{2k}(f_1,f_2,D,\mathcal{B}) = \tfrac{(-1)^k}{k!}\zeta(-k,f_1,f_2,D,\mathcal{B}), \tag{6}$$

$$\beta_{2k+1}(f_1,f_2,D,\mathcal{B}) = \Gamma(-k-\tfrac{1}{2})\text{Res }\zeta(-k-\tfrac{1}{2},f_1,f_2,D,\mathcal{B}). \tag{7}$$

We now proceed with the explicit calculation of the heat content asymptotics on the ball. The ability to perform a special case calculation strongly depends on the choice of the initial temperature f_1 and of the specific heat f_2. We establish Lemma 3 giving the non-locality of the heat content asymptotics by considering the functions:

$$f_i^{(1)}=f^{(1)}=\begin{pmatrix} 0 \\ Z_+^{(0)}(\Omega) \end{pmatrix} \quad \text{and} \quad f_i^{(2)}=f^{(2)}=\begin{pmatrix} rZ_+^{(0)}(\Omega) \\ 0 \end{pmatrix}.$$

These spinors have the property $\Pi f^{(1)}=f^{(1)}$, whereas $\Pi f^{(2)}=0$. Furthermore, because the spinor spherical harmonis involved are orthogonal,

$$(f^{(j)},\varphi_\pm^{(-)})_{L^2}=0.$$

To evaluate the scalar product with $\varphi_\pm^{(+)}$, the relevant r-integrals are [14]

$$\textstyle\int_0^1 dx\ x^{\nu+1}J_\nu(\mu x)=\frac{1}{\mu}J_{\nu+1}(\mu).$$

We first proceed with $f^{(1)}$. We have

$$(f^{(1)},\varphi_\pm^{(+)})_{L^2}=\pm\tfrac{1}{J_{m/2}(\mu)}\textstyle\int_0^1 dr\ r^{m/2}J_{m/2-1}(\mu r)=\pm\frac{1}{\mu}.$$

Let the contour γ enclose all positive zeroes of $J_{m/2-1}(k)$. In the contour integral formalism developed in [4, 5], we have the following representation:

$$\zeta(s,f^{(1)},f^{(1)},D,\mathcal{B})=-2\textstyle\int_\gamma \frac{dk}{2\pi i}k^{-2s-2}\frac{\partial}{\partial k}\ln J_{m/2-1}(k).$$

Deforming the contour towards the imaginary axis we arrive at

$$
\begin{aligned}
\zeta(s,f^{(1)},f^{(1)},D,\mathcal{B}) &= 2\textstyle\int_{\gamma_\epsilon}\frac{dk}{2\pi i}k^{-2s-2}\frac{\partial}{\partial k}\ln J_{m/2-1}(k) \\
&\quad -2\frac{\sin(\pi s)}{\pi}\textstyle\int_\epsilon^\infty dk\ k^{-2s-2}\frac{\partial}{\partial k}\ln I_{m/2-1}(k),
\end{aligned}
$$

with $0 < \epsilon \in \mathbb{R}$ smaller than the first positive zero of $J_{m/2-1}(k)$ and with a semicircle around zero of radius ϵ in the right half plane:

$$\gamma_\epsilon = \{\epsilon e^{it}, t \in [\pi/2, -\pi/2]\}.$$

We determine the contributions of the two terms separately. Although each term depends on ϵ we know the sum will be independent of ϵ and for that reason we concentrate on the ϵ-independent part of both terms. The relevant information to recover the properties in (6) and (7) is encoded in the small-k behavior of $J_{m/2-1}(k)$. The full expansion is [14]

$$J_\nu(k) = \left(\frac{k}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)} \sum_{l=0}^\infty (-1)^l \frac{\Gamma(\nu+1)}{l!\Gamma(\nu+l+1)} \left(\frac{k}{2}\right)^{2l},$$

from which we may obtain the expansion:

$$\ln J_\nu(k) = \nu \ln k - \ln\left[2^\nu \Gamma(\nu+1)\right] + \sum_{l=1}^\infty g_l k^{2l}.$$

Hereby the coefficients g_l are defined. In particular $g_1 = -1/[4(\nu+1)]$. It is easy to see that from the small-ϵ circle no contribution to the residues results, but that

$$\zeta_\epsilon(0, f^{(1)}, f^{(1)}, D, \mathcal{B}) = \tfrac{1}{m}, \quad \zeta_\epsilon(-1, f^{(1)}, f^{(1)}, D, \mathcal{B}) = -\tfrac{m}{2} + 1.$$

We consider next the contribution along the imaginary axis. This time the large-k behavior is needed to determine the information in equations (6) and (7). For large k, the relevant expansion of the Bessel function is [14]

$$I_\nu(k) \sim \frac{e^k}{\sqrt{2\pi k}} \sum_{l=0}^\infty \frac{(-1)^l}{(2k)^l} \frac{\Gamma(\nu+1/2+l)}{l!\Gamma(\nu+1/2-l)}$$

and we define coefficients h_j by

$$\ln I_\nu(k) \sim k - \frac{1}{2}\ln(2\pi k) + \sum_{j+1}^\infty h_j k^{-j}.$$

The needed k-integrals are trivial, $\int_\epsilon^\infty dx\, x^{-\alpha} = \frac{\epsilon^{1-\alpha}}{\alpha-1}$, and the ϵ-independent pieces at the particular values of s needed are easily obtained. We have that:

$$\begin{aligned}
\zeta(0, f^{(1)}, f^{(1)}, D, \mathcal{B}) &= 0, \\
\text{Res}\,(-\tfrac{1}{2}, f^{(1)}, f^{(1)}, D, \mathcal{B}) &= \tfrac{1}{\pi}, \\
\zeta(-1, f^{(1)}, f^{(1)}, D, \mathcal{B}) &= -\tfrac{m-1}{2}, \\
\text{Res}\,(-k-\tfrac{1}{2}, f^{(1)}, f^{(1)}, D, \mathcal{B}) &= \pi^{-1}(-1)^{k+1}(2k-1)h_{2k-1}, \quad k \in \mathbb{N}, \\
\zeta(-k, f^{(1)}, f^{(1)}, D, \mathcal{B}) &= (-1)^k 2(k-1)h_{2k-2}, \quad k-1 \in \mathbb{N}.
\end{aligned}$$

For the heat content coefficients we conclude from (6) and (7) that

$$\begin{aligned}
\beta_0(f^{(1)}, f^{(1)}, D, \mathcal{B}) &= \tfrac{1}{m}, \quad \beta_1(f^{(1)}, f^{(1)}, D, \mathcal{B}) = -\tfrac{2}{\sqrt{\pi}}, \\
\beta_2(f^{(1)}, f^{(1)}, D, \mathcal{B}) &= \tfrac{m-1}{2},
\end{aligned}$$

which is in agreement with

$$\begin{aligned}\beta_0(f^{(1)},f^{(1)},D,\mathcal{B}) &= \int_{\mathcal{M}} dx\ \langle f^{(1)},f^{(1)}\rangle,\\ \beta_1(f^{(1)},f^{(1)},D,\mathcal{B}) &= -2\pi^{-1/2}\int_{\partial\mathcal{M}} dy\ \langle \Pi f^{(1)},\Pi f^{(1)}\rangle,\\ \beta_2(f^{(1)},f^{(1)},D,\mathcal{B}) &= \frac{1}{2}\int_{\partial\mathcal{M}} dy\ L_{aa}\langle \Pi f^{(1)},\Pi f^{(1)}\rangle,\end{aligned}$$

and whereby $c_0=-1$ and $c_2=\frac{1}{2}$ has been determined; thus completing the proof of Lemma 5. As a further check that it is really the projection Π that enters the coefficients we next consider the function $f^{(2)}$. We have $(f^{(2)},\varphi_\pm^{(+)})_{L^2}=\frac{iJ_{m/2+1}}{kJ_{m/2}}$. So the starting point for the associated zeta function is

$$\zeta(s,f^{(2)},f^{(2)},D,\mathcal{B})=2\int_\gamma \frac{dk}{2\pi i}k^{-2s-2}\frac{J^2_{m/2+1}(k)}{J^2_{m/2}(k)}\frac{\partial}{\partial k}\ln J_{m/2-1}(k). \tag{8}$$

Note that the contour γ encloses the zeroes of $J_{m/2-1}(k)$ only. In fact, it is possible to place the contour such that no zeroes of $J_{m/2}(k)$ are enclosed, because the zeroes of $J_{m/2-1}(k)$ are simple, thus $0\neq J'_{m/2-1}(k)=-J_{m/2}(k)$.

Next we use [14]

$$\begin{aligned}kJ_{m/2+1}(k) &= mJ_{m/2}(k)-kJ_{m/2-1}, \qquad \text{so}\\ k^2J^2_{m/2+1}(k) &= m^2J^2_{m/2}(k)+k^2J^2_{m/2-1}-2kmJ_{m/2}(k)J_{m/2-1}(k).\end{aligned}$$

We use the residue theorem, to see only the first term can contribute in (8). Thus

$$\zeta(s,f^{(2)},f^{(2)},D,\mathcal{B})=2m^2\int_\gamma\frac{dk}{2\pi i}k^{-2s-4}\frac{\partial}{\partial k}\ln J_{m/2-1}(k).$$

Proceeding as before, we find

$$\begin{aligned}&\zeta(0,f^{(2)},f^{(2)},D,\mathcal{B})=\tfrac{1}{m+2}, &&\text{Res }\zeta(-1/2,f^{(2)},f^{(2)},D,\mathcal{B})=0,\\ &\zeta(-1,f^{(2)},f^{(2)},D,\mathcal{B})=m, &&\text{so}\quad \beta_0(f^{(2)},f^{(2)},D,\mathcal{B})=\tfrac{1}{m+2},\\ &\beta_1(f^{(2)},f^{(2)},D,\mathcal{B})=0, &&\text{and}\quad \beta_2(f^{(2)},f^{(2)},D,\mathcal{B})=-m.\end{aligned}$$

This is consistent the form given in Ansatz 4. No boundary contributions are found as a result of $\Pi f^{(2)}=0$.

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