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A LOCAL EXISTENCE THEOREM FOR THE EINSTEIN-DIRAC EQUATION

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ABSTRACT. We sketch the proof that, for any manifold M^n admitting real Killing spinors (resp. parallel spinors), there exist warped product metrics $\overline{\eta}$ on $M^n \times \mathbb{R}$ such that $(M^n \times \mathbb{R}, \overline{\eta})$ admit solutions to the Einstein-Dirac equation (resp. the weak Killing equation). The key idea is to split the Einstein-Dirac equation into evolution equations and constraints, by means of Cartan's frame formalism, and to apply the local preservation property of constraints.

1. Introduction

Let (P^m, η) be an m-dimensional smooth oriented Riemannian spin manifold and denote by Ric and S the Ricci tensor and the scalar curvature, respectively. Let $(\, , \,) := \mathrm{Re}\langle \, , \, \rangle$ be the real part of the standard Hermitian product $\langle \, , \, \rangle$ on the spinor bundle $\Sigma(P)$ over P^m. Let D be the Dirac operator acting on sections $\psi \in \Gamma(\Sigma(P))$ of the spinor bundle $\Sigma(P)$. The Einstein-Dirac equation is a minimal coupling of the Dirac equation to the Einstein equation and defined by (see [11])

$$D\psi = \lambda\psi, \qquad \mathrm{Ric} - \frac{1}{2}S\eta = T,$$

where $\lambda \in \mathbb{R}$ is some real number and the energy-momentum tensor T is given by

$$T(X,Y) = \pm\frac{1}{4}(X \cdot \nabla_Y\psi + Y \cdot \nabla_X\psi, \ \psi).$$

A non-trivial spinor field ψ solving this Einstein-Dirac system is called an *Einstein spinor* to eigenvalue $\lambda \in \mathbb{R}$. In case that the scalar curvature S does not vanish at any point, one defines the *weak Killing equation* by

$$\nabla_X\psi = \frac{m}{2(m-1)S}dS(X)\psi + \frac{1}{2(m-1)S}X \cdot dS \cdot \psi + \frac{2\lambda}{(m-2)S}\mathrm{Ric}(X) \cdot \psi - \frac{\lambda}{m-2}X \cdot \psi,$$

where $\lambda \in \mathbb{R}$ is some real number. A non-trivial solution ψ to the equation is called a *weak Killing spinor* to weak Killing number λ (shortly, WK-spinor to WK-number λ). Since rescaling the length of any WK-spinor provides an Einstein spinor, the

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WK-equation is stronger than the Einstein-Dirac equation (In dimension $n = 3$, the considered two equations are essentially equivalent). Moreover, the WK-equation reduces to the Killing equation [3,8],

$$\nabla_X \psi = -\frac{\lambda}{m} X \cdot \psi,$$

if the metric η is Einstein.

Till now, the known examples of the Einstein spinors on Riemannian manifolds are as follows:

(i) Real Killing spinors [2,3,9,15].

(ii) WK-spinors on quasi-Einstein Sasakian manifolds [11].

(iii) Einstein spinors on product manifolds $M^6 \times N^r$, where M^6 is a six-dimensional simply connected nearly Kähler manifold and N^r is a manifold of general dimension r admitting Killing spinors [11].

(iv) WK-spinors on the three-dimensional sphere S^3 with non-standard merics [4,10,11].

(v) WK-spinors on the three-dimensional Euclidean space $\mathbb{R}^3$ with non-constant scalar curvature [11,13].

The aim of this article is to sketch the ideas behind the proof of the following special existence theorem for WK- as well as Einstein spinors [12].

Main Theorem: *Let (M^n, g_M) be a Riemannian manifold admitting a real Killing spinor ψ_M. Then, for any real number $\lambda_Q \in \mathbb{R}$, there exists a warped product metric $\overline{\eta}$ on $Q^{n+1} = M^n \times \mathbb{R}$ such that $(Q^{n+1}, \overline{\eta})$ admits an Einstein spinor ψ to eigenvalue λ_Q. In particular, if ψ_M is a parallel spinor, then the Einstein spinor ψ becomes a WK-spinor to WK-number λ_Q.*

To prove the theorem we will split the Einstein-Dirac equation into evolution equations and constraints, by means of Cartan's frame formalism, and apply the local preservation property of the constraints. In fact, we consider an initial-value formulation for the Einstein-Dirac equation, in Riemannian setting, and solve it for a specific class of initial data sets. It is well-known that, in Riemannian signature the Einstein equations are generally of elliptic type, making the initial-value problem (the Cauchy problem) for general smooth data inappropriate. However, when the considered Riemannian manofolds admit a codimension one foliation, one can represent the Einstein equations to be of hyperbolic type, just as one does over Lorentzian manifolds, and can indeed formulate the initial-value problem in a natural way. So far, not much has been studied about the initial-value problem for the Einstein-Dirac equation. In Lorentzian signature, the spacelike initial-value problem for the Einstein-Dirac system was considered by Bao/Isenberg/Yasskin [1] in terms of 3+1 Hamiltonian formalism, but no existence theorem was proved there. Recently, Friedrich/Rendall indicated [7], in terms of Penrose's two-spinor formalism, that the Einstein-Dirac equation may be reduced to symmetric hyperbolic evolution equations, illustrating some questions arising in the reduction.

2. ADM-representation of metrics

In this section, we give a brief description of the Riemannian setting of the so-called ADM-representation of metrics. This representation enables us to transform the Einstein equations to hyperbolic systems in PDE theory (see the evolution equations (E1)-(E3) in the next section)

Let (Q^{n+1}, η) be an (n+1)-dimensional smooth oriented Riemannian spin manifold. We assume that there exists a codimension one foliation on (Q^{n+1}, η) defined by a unit vector field E_{n+1} with $dE^{n+1} = 0$, where $E^{n+1} = \eta(E_{n+1}, \cdot)$ is the dual 1-form of E_{n+1}. Let $E_{n+1}^{\perp}$ denote the η-orthogonal complement of E_{n+1} in the tangent bundle $T(Q)$. Let $(E_1, \dots, E_n, E_{n+1})$ be a local η-orthonormal frame field on Q^{n+1}, with $E_j \in E_{n+1}^{\perp}, j = 1, \cdots, n$, and $(E^1, \dots, E^n, E^{n+1})$ the dual frame field. Denote $\otimes_s^r(E_{n+1}^{\perp})$ the space of all (r,s)-tensor fields B on Q^{n+1} such that

$$\eta(E_{i_1} \otimes \cdots \otimes E_{i_r},\ B(E_{j_1} \otimes \cdots \otimes E_{j_s})) = 0,$$

whenever either $i_k = n+1$ for some i_k or $j_l = n+1$ for some j_l.

Now, consider a positive definite (1,1)-tensor field K on (Q^{n+1}, η). Letting $\overline{\eta}$ be the metric induced by K via $\overline{\eta}(X,Y) = \eta(K(X), K(Y))$ and identifying $\overline{\eta}$ with K^2, we can express $\overline{\eta}$ as

$$\overline{\eta} = K^2 = \Big\{ \sum_{i,j=1}^{n} (L^2)^i_j E^j \otimes E_i \Big\} + E^{n+1} \otimes L^2(\zeta) + \eta(L^2(\zeta), \cdot\,) \otimes E_{n+1}$$
$$+ \Big\{ \eta(L(\zeta), L(\zeta)) + \rho^2 \Big\} E^{n+1} \otimes E_{n+1},$$

where $L \in \otimes_1^1(E_{n+1}^{\perp})$, $\zeta \in \otimes_0^1(E_{n+1}^{\perp})$ and $\rho : Q^{n+1} \longrightarrow \mathbb{R}$ is a positive function. This is an invariant Riemannian version of the well-known ADM-representation of metrics in general relativity. ζ agrees with the *shift vector field* and ρ with the *lapse function*. Note that L^2 is positive-definite on each slice of the foliation and, on the slices, coincides with the metrics induced by K^2. Certainly,

$$\Big(F_1 := L^{-1}(E_1), \dots, F_n := L^{-1}(E_n), F_{n+1} := \rho^{-1}(E_{n+1} - \zeta) \Big)$$

is a local $\overline{\eta}$-orthonormal frame field on Q^{n+1}, its dual frame field being given by

$$F^i = L(E^i) + \eta(L(\zeta), E_i) E^{n+1}, \qquad F^{n+1} = \rho E^{n+1}.$$

For any vector field V in $E_{n+1}^{\perp}$, we have $\overline{\eta}(V, F_{n+1}) = 0$ and so $E_{n+1}^{\perp}$ coincides with the $\overline{\eta}$-orthogonal complement of F_{n+1} in $T(Q)$.

In the following, we fix some notations. Let

$$\mathrm{II}(V) := -\nabla^{\overline{\eta}}_V F_{n+1} \qquad \text{and} \qquad \Theta(V) := -\nabla^{\eta}_V E_{n+1}$$

denote the second fundamental form, on each slice, defined by the unit vector field F_{n+1} and E_{n+1}, respectively. Let $\overline{g}$ (resp. g) denote the metric, on each slice, induced by $\overline{\eta}$ (resp. η) and $\nabla^{\overline{g}}$ (resp. ∇^g) its Levi-Civita connection. In the notations one can represent the Ricci tensor $\mathrm{Ric}_{\overline{\eta}}$ hyperbolically as follows.

Proposition 2.1.

$$
\begin{aligned}
&(\nabla^{\eta}_{E_{n+1}} II)(V,W) \\
&= \rho \cdot \Big\{ \ Ric_{\overline{\eta}}(V,W) - \ Ric_{\overline{g}}(V,W) - 2\overline{\eta}(V, II^2(W)) + (Tr_{\overline{g}} II) II(V,W) \Big\} \\
&\quad + (\nabla^{\overline{g}}_{\zeta} II)(V,W) + \overline{\eta}(II(V), \nabla^{\overline{g}}_{W} \zeta) + \overline{\eta}(II(W), \nabla^{\overline{g}}_{V} \zeta) \\
&\quad + \overline{\eta}(\Theta(V), II(W)) + \overline{\eta}(\Theta(W), II(V)) + (\nabla^{\overline{g}}_{W} d\rho)(V),
\end{aligned}
$$

where V, W are vector fields in $E^{\perp}_{n+1}$.

Throughout the article we will identify $\Sigma(Q)_{\overline{\eta}}$ with $\Sigma(Q)_{\eta}$ and $\psi \in \Gamma(\Sigma(Q)_{\overline{\eta}})$ with its pullback $\widetilde{K}(\psi)$, via the natural isomorphism $\widetilde{K} : \Sigma(Q)_{\overline{\eta}} \longrightarrow \Sigma(Q)_{\eta}$, and write simply as $\Sigma(Q)$ and ψ, respectively. In the notations, the spin derivatives $\nabla^{\overline{\eta}}\psi$, $\nabla^{\overline{g}}\psi$ are related, on Q^{n+1}, by

$$\nabla^{\overline{\eta}}_{V}\psi = \nabla^{\overline{g}}_{V}\psi + \frac{1}{2}\mathrm{II}(V) \cdot F_{n+1} \cdot \psi$$

and the Dirac equation becomes

$$\lambda_Q \psi = D_{\overline{\eta}}\psi = \sum_{i=1}^{n} F_i \cdot \nabla^{\overline{g}}_{F_i}\psi - \frac{1}{2}(\mathrm{Tr}_{\overline{g}}\mathrm{II}) F_{n+1} \cdot \psi + F_{n+1} \cdot \nabla^{\overline{\eta}}_{F_{n+1}}\psi.$$

The hyperbolic representation of the Dirac equation is explicitly given in the next section (see (E3)).

3. The initial-value problem for the Einstein-Dirac equation

We set up an invariant initial-value formulation for the Einstein-Dirac equation

$$\mathrm{Ric}_{\overline{\eta}} - \frac{1}{2} S_{\overline{\eta}}\overline{\eta} = T_{\overline{\eta}}, \qquad D_{\overline{\eta}}\psi = \lambda_Q \psi, \quad \lambda_Q \in \mathbb{R},$$

where

$$T_{\overline{\eta}}(X,Y) = \frac{\epsilon}{4}\Big(X \cdot \nabla^{\overline{\eta}}_{Y}\psi + Y \cdot \nabla^{\overline{\eta}}_{X}\psi, \ \psi \Big), \qquad \epsilon = \pm 1.$$

The formulation will be applied in the next section to establish a local existence theorem for a specific class of initial data sets. Following the work [7] as a guideline, we can indeed express the evolution equations as well as the constraints in an invariant form. For simplicity, we write

$$\Delta := \mathrm{Ric}_{\overline{\eta}} - \frac{1}{2} S_{\overline{\eta}}\overline{\eta} - T_{\overline{\eta}}.$$

The tensor field Δ decomposes into three parts

$$\Delta = \Delta^{E} + \Big\{ \Delta^{M} \otimes F^{n+1} + F^{n+1} \otimes \Delta^{M} \Big\} + \Delta^{H}\Big(F^{n+1} \otimes F^{n+1} \Big),$$

where

$$\Delta^{E} = \sum_{i,j=1}^{n} \Delta(F_i, F_j) F^i \otimes F^j, \quad \Delta^{M} = \sum_{i=1}^{n} \Delta(F_{n+1}, F_i) F^i, \quad \Delta^{H} = \Delta(F_{n+1}, F_{n+1}).$$

Restricting the equations, $\Delta^M = 0$ and $\Delta^H = 0$, to a fixed slice, we obtain the *momentum constraint*

$$T_{\overline{g}}(F_{n+1}, V) = d(\mathrm{Tr}_{\overline{g}}\mathrm{II})(V) - \mathrm{div}_{\overline{g}}(\mathrm{II})(V)$$

and the *Hamiltonian constraint*

$$T_{\overline{g}}(F_{n+1}, F_{n+1}) = -\frac{1}{2}S_{\overline{g}} + \frac{1}{2}(\mathrm{Tr}_{\overline{g}}\mathrm{II})^2 - \frac{1}{2}\mathrm{Tr}_{\overline{g}}(\mathrm{II}^2),$$

where $T_{\overline{g}}$ denotes the restriction of $T_{\overline{\eta}}$ to the slices. The information on the evolution should be contained in $\Delta^E = 0$, or any combination of it with the constraints. The evolution equations should be chosen in such a way that, under the evolution, local preservation of the constraints is guaranteed. We consider the evolution equations of the form,

$$\Delta(V, W) = \Delta(F_{n+1}, F_{n+1}) \cdot \overline{\eta}(V, W) \quad \text{and} \quad D_{\overline{\eta}}\psi = \lambda_Q \psi, \quad \lambda_Q \in \mathbb{R}.$$

Then, it turns out that the evolution system consists of three differential equations, describing the evolution of metrics $L^2 = \overline{g}$, that of symmetric (0,2)-tensor fields II and that of spinor fields ψ , respectively:

(E1)
$$\begin{aligned}
&\eta((\nabla^{\eta}_{E_{n+1}}L)(V),\ L(W)) + \eta(L(V),\ (\nabla^{\eta}_{E_{n+1}}L)(W))\\
&= \eta((\nabla^{g}_{\zeta}L)(V),\ L(W)) + \eta(L(V),\ (\nabla^{g}_{\zeta}L)(W))\\
&\quad + \eta(\nabla^{g}_{V}\zeta + \Theta(V) - \rho\mathrm{II}(V),\ L^2(W)) + \eta(L^2(V),\ \nabla^{g}_{W}\zeta + \Theta(W) - \rho\mathrm{II}(W)),
\end{aligned}$$

(E2)
$$\begin{aligned}
&(\nabla^{\eta}_{E_{n+1}}\mathrm{II})(V, W)\\
&= \frac{\rho}{n-1}\Big\{S_{\overline{g}} + \mathrm{Tr}_{\overline{g}}(\mathrm{II}^2) - (\mathrm{Tr}_{\overline{g}}\mathrm{II})^2\Big\}\overline{g}(V, W)\\
&\quad + \frac{\epsilon\rho}{4}\Big(V \cdot \nabla^{\overline{g}}_{W}\psi + W \cdot \nabla^{\overline{g}}_{V}\psi,\ \psi\Big)\\
&\quad + \frac{\epsilon\rho}{8}\Big(\{V \cdot \mathrm{II}(W) + W \cdot \mathrm{II}(V)\} \cdot F_{n+1} \cdot \psi,\ \psi\Big)\\
&\quad + \frac{\epsilon\lambda_Q\rho}{2(n-1)}(\psi,\ \psi)\overline{g}(V, W) - \frac{\epsilon\rho}{n-1}\Big(\sum_{i=1}^{n} F_i \cdot \nabla^{\overline{g}}_{F_i}\psi,\ \psi\Big)\overline{g}(V, W)\\
&\quad + \rho \cdot \Big\{-\ \mathrm{Ric}_{\overline{g}}(V, W) - 2\overline{g}(V, \mathrm{II}^2(W)) + (\mathrm{Tr}_{\overline{g}}\mathrm{II}) \cdot \mathrm{II}(V, W)\Big\}\\
&\quad + (\nabla^{\overline{g}}_{\zeta}\mathrm{II})(V, W) + \overline{g}(\mathrm{II}(V), \nabla^{\overline{g}}_{W}\zeta) + \overline{g}(\mathrm{II}(W), \nabla^{\overline{g}}_{V}\zeta)\\
&\quad + \overline{g}(\Theta(V), \mathrm{II}(W)) + \overline{g}(\Theta(W), \mathrm{II}(V)) + (\nabla^{\overline{g}}_{W}d\rho)(V),
\end{aligned}$$

(E3)

$$
\begin{aligned}
&\nabla^{\eta}_{E_{n+1}}\psi \\
=\ &\nabla^{g}_{\zeta}\psi - \lambda_Q \rho E_{n+1}\cdot\psi + \frac{\rho}{2}(\mathrm{Tr}_{\overline{g}}\mathrm{II})\psi \\
&+ \rho E_{n+1}\cdot\Big\{\sum_{i=1}^{n} E_i\cdot\nabla^{g}_{L^{-1}E_i}\psi + \frac{1}{4}\sum_{j,k,l=1}^{n}(\Lambda_g)_{jkl}E_j\cdot E_k\cdot E_l\cdot\psi\Big\} \\
&+ \frac{1}{4}\sum_{i=1}^{n} E_i\cdot(\nabla^{\eta}_{E_{n+1}}L)(L^{-1}E_i)\cdot\psi + \frac{\rho}{4}\sum_{i=1}^{n}E_i\cdot(L\circ\mathrm{II}\circ L^{-1})(E_i)\cdot\psi \\
&- \frac{1}{4}\sum_{i=1}^{n}E_i\cdot(\nabla^{g}_{\zeta}L)(L^{-1}E_i)\cdot\psi - \frac{1}{4}\sum_{i=1}^{n}E_i\cdot L(\nabla^{g}_{L^{-1}E_i}\zeta)\cdot\psi \\
&- \frac{1}{4}\sum_{i=1}^{n}E_i\cdot(L\circ\Theta\circ L^{-1})(E_i)\cdot\psi - \frac{1}{2}\sum_{i=1}^{n}d\rho(L^{-1}E_i)E_i\cdot E_{n+1}\cdot\psi.
\end{aligned}
$$

Deriving the constraint equations on initial hypersurfaces, one can also define the initial data sets as follows.

Definition 3.1 (In case of $n = 2m$) An *initial data set* $(M^{2m}, \overline{g}, \mathrm{II}_M, \psi_M)$ for the Einstein-Dirac equation on Q^{2m+1} consists of a slice M^{2m} with, defined on it, a metric $\overline{g}$, a symmetric $(0,2)$-tensor field II_M and a spinor field ψ_M satisfying the *momentum constraint*

$$
\begin{aligned}
&d(\mathrm{Tr}_{\overline{g}}\mathrm{II}_M)(V) - \mathrm{div}_{\overline{g}}(\mathrm{II}_M)(V) \\
=&\frac{\epsilon}{4}\Big((\sqrt{-1})^{m+1}\mu_{\overline{g}}\cdot\Big\{\nabla^{\overline{g}}_V\psi_M - V\cdot D_{\overline{g}}\psi_M\Big\},\ \psi_M\Big), \quad \epsilon = \pm 1,
\end{aligned}
$$

as well as the *Hamiltonian constraint*

$$
-S_{\overline{g}} + (\mathrm{Tr}_{\overline{g}}\mathrm{II}_M)^2 - \mathrm{Tr}_{\overline{g}}(\mathrm{II}_M\circ\mathrm{II}_M) = -\epsilon\Big(D_{\overline{g}}\psi_M - \lambda_Q\psi_M,\ \psi_M\Big), \quad \lambda_Q\in\mathbb{R}.
$$

Definition 3.2 (In case of $n = 2m-1$) An *initial data set* $(M^{2m-1}, \overline{g}, \mathrm{II}_M, \psi^{+}_M, \varphi^{+}_M)$ for the Einstein-Dirac equation on Q^{2m} consists of a slice M^{2m-1} with, defined on it, a metric $\overline{g}$, a symmetric $(0,2)$-tensor field II_M and two spinor fields $\psi^{+}_M, \varphi^{+}_M$ satisfying the *momentum constraint*

$$
\begin{aligned}
&d(\mathrm{Tr}_{\overline{g}}\mathrm{II}_M)(V) - \mathrm{div}_{\overline{g}}(\mathrm{II}_M)(V) \\
=&\frac{\epsilon}{4}\Big(\nabla^{\overline{g}}_V\psi^{+}_M - V\cdot D_{\overline{g}}\psi^{+}_M,\ \varphi^{+}_M\Big) - \frac{\epsilon}{4}\Big(\nabla^{\overline{g}}_V\varphi^{+}_M - V\cdot D_{\overline{g}}\varphi^{+}_M,\ \psi^{+}_M\Big)
\end{aligned}
$$

as well as the *Hamiltonian constraint*

$$
\begin{aligned}
&- S_{\overline{g}} + (\mathrm{Tr}_{\overline{g}}\mathrm{II}_M)^2 - \mathrm{Tr}_{\overline{g}}(\mathrm{II}_M\circ\mathrm{II}_M) \\
=&-\epsilon\Big(D_{\overline{g}}\psi^{+}_M,\ \varphi^{+}_M\Big) - \epsilon\Big(D_{\overline{g}}\varphi^{+}_M,\ \psi^{+}_M\Big) + \epsilon\lambda_Q\Big\{(\psi^{+}_M,\psi^{+}_M) + (\varphi^{+}_M,\varphi^{+}_M)\Big\}, \quad \lambda_Q\in\mathbb{R}.
\end{aligned}
$$

4. A local existence theorem

For a specific class of initial data sets, we establish a local existence theorem for the Einstein-Dirac equation. We consider here only the case $n = 2m$, the other case $n = 2m-1$ can be treated in a similar way. Let $Q^{2m+1} = M^{2m} \times \mathbb{R}$ be a product manifold, and let the product metric $\eta = g_M \times g_{\mathbb{R}}$ be the reference metric on Q^{2m+1}, where g_M indicates an arbitrary Riemannian metric on M^n and $g_{\mathbb{R}}$ the standard metric on the real line $\mathbb{R}$. We write $g_{\mathbb{R}} = dt \otimes dt$, using the standard coordinate $t \in \mathbb{R}$. By $(E_1, \dots, E_n)$ we denote a local orthonormal frame on (M^{2m}, g_M) as well as its lift to (Q^{2m+1}, η). Let $E_{2m+1} = \frac{d}{dt}$ denote the unit vector field on $(\mathbb{R}, g_{\mathbb{R}})$ as well as the lift to (Q^{2m+1}, η). For simplicity, we denote by $\mathrm{WP}(g_M; a)$ the following class of metrics (the *warped products of g_M and $g_{\mathbb{R}}$*) :

$$\overline{\eta} = e^f \Big(\sum_{i=1}^{2m} E^i \otimes E^i \Big) + e^{af} dt \otimes dt,$$

where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a real-valued function and $a \in \mathbb{R}$ is a real number. Let $\psi_M = \psi_M^+ + \psi_M^-$ be a spinor field on (M^{2m}, g_M) with $\psi_M^{\pm} \in \Gamma(\Sigma^{\pm}(M))$, where $\Sigma^+(M)$(resp. $\Sigma^-(M)$) indicates the positive (resp. negative) part of $\Sigma(M)$. Let $\Gamma_{\mathrm{odd}}(\psi_M)$ denote the space of all spinor fields of the form $\psi = h^+\psi_M^+ + h^-\psi_M^-$ on $Q^{2m+1} = M^{2m} \times \mathbb{R}$ defined by

$$\psi(x,t) = h^+(t)\psi_M^+(x) + h^-(t)\psi_M^-(x), \qquad (x,t) \in M^{2m} \times \mathbb{R},$$

where $h^{\pm} : \mathbb{R} \longrightarrow \mathbb{C}$ are complex-valued functions.

Proposition 4.1. *Let $\psi_M = \psi_M^+ + \psi_M^-$ be a real Killig spinor on (M^{2m}, g_M) with*

$$\nabla_V^{g_M} \psi_M^{\pm} = -\frac{\lambda_M}{2m} V \cdot \psi_M^{\mp}, \quad \lambda_M \in \mathbb{R}.$$

For $\overline{\eta} \in WP(g_M; a)$ and $\psi \in \Gamma_{\mathrm{odd}}(\psi_M)$, the evolution equations (E1), (E2), (E3) for the Einstein-Dirac equation are equivalent to

(E1-E2)

$$\begin{aligned} f_{tt} = {} & \frac{a f_t f_t}{2} - \frac{2}{m^2}(\lambda_M)^2 e^{(a-1)f} - \frac{\epsilon\lambda_Q}{2m-1} e^{af} \langle h^+, h^+\rangle (\psi_M, \psi_M) \\ & + \frac{2m+1}{4m(2m-1)} \epsilon \lambda_M e^{(a-\frac{1}{2})f} \{\langle h^+, h^-\rangle + \langle h^-, h^+\rangle\} (\psi_M, \psi_M) \end{aligned}$$

and

(E3)

$$\begin{aligned} h_t^+ &= -\frac{m}{2} f_t h^+ + (\sqrt{-1})^{2m+3} \lambda_Q e^{\frac{a}{2}f} h^+ - (\sqrt{-1})^{2m+3} \lambda_M e^{\frac{1}{2}(a-1)f} h^-, \\ h_t^- &= (\sqrt{-1})^{2m+3} \lambda_M e^{\frac{1}{2}(a-1)f} h^+ - \frac{m}{2} f_t h^- - (\sqrt{-1})^{2m+3} \lambda_Q e^{\frac{a}{2}f} h^-, \end{aligned}$$

where we have used the notation $\langle\, , \rangle$ (for complex-valued functions) to mean the standard Hermitian product.

Let $\mathrm{Re}(h^{\pm})$ and $\mathrm{Im}(h^{\pm})$ denote the real and imaginary part of the complex-valued functions $h^{\pm}$, respectively. Then, we observe that, if we take

$$\Psi = \Big(f,\, f_t,\, \mathrm{Re}(h^+),\, \mathrm{Im}(h^+),\, \mathrm{Re}(h^-),\, \mathrm{Im}(h^-)\Big)$$

as a set of six unknowns, then the system of evolution equations in Proposition 4.1 reduces to an autonomous eqution

$$\frac{d}{dt}\Psi = H(\Psi)$$

for some vector field H defined on the six-dimensional Euclidean space $\mathbb{R}^6$. This implies that, to each initial data, there corresponds a unique smooth local solution to the evolution system in Proposition 4.1.

Proposition 4.2. *Let (M^{2m}, g_M) be a Riemannian manifold admitting a real Killing spinor ψ_M. Then, for any real number $\lambda_Q \in \mathbb{R}$, there exists an open interval $(-\omega, \omega) \subset \mathbb{R}$ and a warped product metric $\overline{\eta}$ on $Q^{2m+1} = M^{2m} \times (-\omega, \omega)$ such that $(Q^{2m+1}, \overline{\eta})$ admits an Einstein spinor ψ to eigenvalue λ_Q. In particular, if ψ_M is a parallel spinor, then the Einstein spinor ψ becomes a WK-spinor.*

This proposition can be extended so as to include the case $n = 2m-1$. Although the Einstein spinor ψ in Proposition 4.2 does not generally extend to the whole real line $\mathbb{R}$, we can conclude, via reparametrization $(-\omega, \omega) \longrightarrow \mathbb{R}$, that there exist indeed global solutions to the Einstein-Dirac equation on $M^n \times \mathbb{R}$.

Theorem 4.1. *Let (M^n, g_M) be a Riemannian manifold admitting a real Killing spinor φ_M. Then, for any real number $\lambda_Q \in \mathbb{R}$, there exists a warped product metric $\overline{\eta}^*$ on $Q^{n+1} = M^n \times \mathbb{R}$ such that $(Q^{n+1}, \overline{\eta}^*)$ admits an Einstein spinor φ to eigenvalue λ_Q. In particular, if φ_M is a parallel spinor, then the Einstein spinor φ becomes a WK-spinor to WK-number λ_Q.*

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