Trends in Mathematics
Information Center for Mathematical Sciences
Volume 5, Number 2, December, 2002, Pages 105–114

THE SPECTRAL GEOMETRY OF THE RIEMANN CURVATURE TENSOR

PETER B. GILKEY[1], RAINA IVANOVA[2], AND TAN ZHANG[2]

Abstract. Let $\mathcal{E}$ be a natural operator associated to the curvature tensor of a pseudo-Riemannian manifold. We study when the spectrum, or more generally the real Jordan normal form, of $\mathcal{E}$ is constant on the natural domain of definition. In particular, we examine the Jacobi operator, the higher order Jacobi operator, the Szabo operator, and the skew-symmetric curvature operator.

1. Introduction

Let (M, g) be a pseudo-Riemannian manifold of signature (p, q) and dimension $m = p + q$. The associated Riemann curvature tensor R has the symmetries:

$$\begin{aligned} &R(x,y,z,w) = R(z,w,x,y) = -R(y,x,z,w),\\ &R(x,y,z,w) + R(y,z,x,w) + R(z,x,y,w) = 0. \end{aligned} \tag{1.1}$$

The covariant derivative ∇R has the symmetries:

$$\begin{aligned} &\nabla R(x,y,z,w;v) = \nabla R(z,w,x,y;v) = -\nabla R(y,x,z,w;v),\\ &\nabla R(x,y,z,w;v) + \nabla R(y,z,x,w;v) + \nabla R(z,x,y,w;v) = 0,\\ &\nabla R(x,y,z,w;v) + \nabla R(x,y,w,v;z) + \nabla R(x,y,v,z;w) = 0. \end{aligned} \tag{1.2}$$

Instead of working in the geometric category, one can also work in a purely algebraic category. If V is a vector space of signature (p, q), then we say that $R \in \otimes^4 V^*$ is an *algebraic curvature tensor* on V if R satisfies the symmetries of equation (1.1) and that $\mathfrak{R} \in \otimes^5 V^*$ is an *algebraic covariant derivative curvature tensor* on V if $\mathfrak{R}$ satisfies the symmetries of equation (1.2).

Such tensors are, in general, very complicated objects to study. There are several endomorphisms $\mathcal{E}$ one can associate to R or to ∇R. One studies $\operatorname{spec}(\mathcal{E})$ (i.e. the set of complex eigenvalues of $\mathcal{E}$) or, more generally, the Jordan normal form of $\mathcal{E}$. In this brief note, we survey some recent results for the Jacobi operator, the higher order Jacobi operator, the skew-symmetric curvature operator (in both the real and complex settings), and the Szabó operator.

We work in the algebraic setting for the moment. Let

$$S^{\pm}(V) := \{v \in V : (v, v) = \pm 1\}$$

2000 *Mathematics Subject Classification.* 53B20.
Key words and phrases. Osserman tensors, Ivanov-Petrova tensors, Szabó tensors, higher order Jacobi operator, pseudo-Riemannian manifold, algebraic curvature tensor, algebraic covariant derivative tensor.
[1]Research partially supported by the NSF (USA) and MPI (Leipzig).
[2]Research supported by the NSF (USA).
Received October 1, 2002.

be the *pseudo-spheres* of unit spacelike $(+)$ and timelike $(-)$ vectors in a vector space V of signature (p,q). Let $\operatorname{Gr}_k(V)$ (resp. $\operatorname{Gr}_k^+(V)$) be the *Grassmannian* of all unoriented (resp. oriented) non-degenerate k dimensional subspaces of V;

$$\operatorname{Gr}_k(V) = \dot{\sqcup}_{r+s=k} \operatorname{Gr}_{r,s}(V) \quad \text{and} \quad \operatorname{Gr}_k^+(V) = \dot{\sqcup}_{r+s=k} \operatorname{Gr}_{r,s}^+(V)$$

may be decomposed as a disjoint union, where $\operatorname{Gr}_{r,s}(V)$ (resp. $\operatorname{Gr}_{r,s}^+(V)$) is the set of unoriented (resp. oriented) non-degenerate subspaces of signature (r,s). We say that a pair (r,s) is *admissible* if $\operatorname{Gr}_{r,s}(V)$ consists of more than a single point, or, equivalently, if $0 \le r \le p$, $0 \le s \le q$, and $1 \le r+s \le m-1$. We say k is admissible if $1 \le k \le m-1$.

Definition 1.1. Let R be an algebraic curvature tensor and let $\mathfrak{R}$ be an algebraic covariant derivative curvature tensor on a vector space V of signature (p,q). Let $R(\cdot,\cdot)$ and $\mathfrak{R}.(\cdot,\cdot)$ be the associated endomorphisms of V defined by the identities:

$$\begin{aligned} &(R(x_1,x_2)x_3,x_4) = R(x_1,x_2,x_3,x_4), \quad \text{and} \\ &(\mathfrak{R}_{x_1}(x_2,x_3)x_4,x_5) = \mathfrak{R}(x_2,x_3,x_4,x_5;x_1). \end{aligned}$$

(1) The *Jacobi operator* $J(x) : y \to R(y,x)x$ is a self-adjoint map of V. One says that R is *spacelike Osserman* or *timelike Osserman* if $\operatorname{spec}(J)$ is constant on $S^+(V)$ or on $S^-(V)$, respectively. Similarly, one says that R is *spacelike Jordan Osserman* or *timelike Jordan Osserman* if the Jordan normal form of J is constant on $S^+(V)$ or on $S^-(V)$, respectively.

(2) Let $\{e_1,...,e_k\}$ be an orthonormal basis for $\pi \in \operatorname{Gr}_k(V)$. The *higher order Jacobi operator* $J(\pi) := \sum_{1\le i\le k}(e_i,e_i)J(e_i)$ is a self-adjoint map of V which is independent of the particular orthonormal basis chosen for π. One says that R is *Osserman of type* (r,s) if $\operatorname{spec}(J)$ is constant on $\operatorname{Gr}_{r,s}(V)$. The notion *Jordan Osserman of type* (r,s) is defined similarly.

(3) Let $\{e_1,e_2\}$ be an oriented orthonormal basis for $\pi \in \operatorname{Gr}_2^+(V)$. The *skew-symmetric curvature operator* $\mathcal{R}(\pi) := R(e_1,e_2)$ is independent of the particular orthonormal basis chosen for π. One says that R is *spacelike IP*, *mixed IP*, or *timelike IP* if $\operatorname{spec}(\mathcal{R})$ is constant on $\operatorname{Gr}_{0,2}^+(V)$, $\operatorname{Gr}_{1,1}^+(V)$, or $\operatorname{Gr}_{2,0}^+(V)$, respectively. The notions *spacelike Jordan IP*, *timelike Jordan IP*, and *mixed Jordan IP* are defined similarly.

(4) The *Szabó operator* $\mathfrak{S}(x) : y \to \mathfrak{R}_x(y,x)x$ is self-adjoint. One says that $\mathfrak{R}$ is *spacelike Szabó* or *timelike Szabó* if $\operatorname{spec}(\mathfrak{S})$ is constant on $S^+(V)$ or on $S^-(V)$, respectively; the notions *spacelike Jordan Szabó* and *timelike Jordan Szabó* are defined similarly.

(5) The eigenvalue 0 plays a distinguished role in this subject. We say that R is *2-nilpotent* if $R(x_1,x_2)R(x_3,x_4) = 0$ for any $x_i \in V$; this implies that R is Osserman, Osserman of type (r,s), and IP. Similarly, $\mathcal{R}$ is said to be *2-nilpotent* if $\mathcal{R}(x_1,x_2,x_3)\mathcal{R}(x_3,x_4,x_5) = 0$ for any $x_i \in V$; this implies that $\mathcal{R}$ is Szabó.

The names Osserman, Szabó, and IP are used because the germinal work for this subject in the Riemannian $(p=0)$ setting was done by Osserman [27], by Szabó [30], and by Ivanov and Petrova [24].

In the geometric context, we shall say that a pseudo-Riemannian manifold (M,g) has a given property *pointwise* if the Riemann curvature tensor R or the covariant derivative ∇R has this property on the tangent space T_PM for every point $P \in M$.

Thus, for example, to say that (M, g) is pointwise spacelike Jordan IP is to mean that R is spacelike Jordan IP on $T_P M$ for every point $P \in M$. In particular, the eigenvalues and the Jordan normal form are allowed to vary from point to point. We omit the qualifier 'pointwise' if the structures are independent of the point in question - i.e. if the spectrum or the Jordan normal form does not vary from point to point of the manifold. This is a subtle, but crucial, distinction.

The following result is proved using analytic continuation - see [11] for details:

Theorem 1.2. *Let R be an algebraic curvature tensor and let $\mathfrak{R}$ be an algebraic covariant derivative curvature tensor on a vector space V of signature (p, q).*

(1) *The following conditions are equivalent and if any is satisfied, then R is said to be an Osserman tensor:*
   (a) *If $p \geq 1$, then R is timelike Osserman;*
   (b) *If $q \geq 1$, then R is spacelike Osserman.*
(2) *The following conditions are equivalent and if any is satisfied, then R is said to be a k-Osserman tensor:*
   (a) *There exists (r, s) with $r + s = k$ so R is Osserman of type (r, s);*
   (b) *R is Osserman of type (r, s) for any admissible (r, s) with $r + s = k$.*
(3) *The following conditions are equivalent and if any is satisfied, then R is said to be an IP tensor:*
   (a) *If $p \geq 2$, then R is timelike IP;*
   (b) *If $p \geq 1$ and if $q \geq 1$, then R is mixed IP;*
   (c) *If $q \geq 2$, then R is spacelike IP.*
(4) *The following conditions are equivalent and if any is satisfied, then $\mathfrak{R}$ is said to be a Szabó tensor:*
   (a) *If $p \geq 1$, then R is timelike Szabó;*
   (b) *If $q \geq 1$, then R is spacelike Szabó.*

By Theorem 1.2, timelike Osserman and spacelike Osserman are equivalent conditions. Since the Jordan normal form determines the eigenvalue structure, spacelike Jordan Osserman or timelike Jordan Osserman implies Osserman. The implications are not reversible. As we shall see in Section 2, there exist tensors which are Osserman but which are neither spacelike Jordan Osserman nor timelike Jordan Osserman. There also exist tensors which are spacelike Jordan Osserman but not timelike Jordan Osserman and tensors which are timelike Jordan Osserman but not spacelike Jordan Osserman. Similar observations hold for the other operators; in the higher signature setting, the eigenvalue structure does not determine the Jordan normal form of a self-adjoint or of a skew-adjoint linear map.

The following family of examples will play a central role in our discussion. We refer to [17] for details.

Definition 1.3. *Let $u \geq 2$ and let $(x, y) := (x_1, ..., x_u, y_1, ..., y_u)$ be the usual coordinates on $\mathbb{R}^{2u}$. Let $\psi = \psi_{ij}(x)$ be a symmetric 2 tensor on $\mathbb{R}^u$. We define a metric g_ψ of neutral signature (u, u) on $\mathbb{R}^{2u}$ by setting:*

$$g_\psi := \textstyle\sum_{1 \leq i \leq u} dx^i \circ dy^i + \sum_{1 \leq i,j \leq u} \psi_{ij}(x) dx^i \circ dx^j.$$

Let $\mathbb{R}^{(a,b)}$ be Euclidean space with the canonical flat metric $g_{a,b}$ of signature (a, b). Give $\mathbb{R}^{2u} \times \mathbb{R}^{(a,b)}$ the product metric of signature $(p, q) = (u + a, u + b)$:

$$g_{\psi,a,b} := g_\psi + g_{a,b} \quad on \quad \mathbb{R}^{2u} \times \mathbb{R}^{(a,b)}.$$

Definition 1.4. *If f is a smooth function on $\mathbb{R}^u$, let $\psi_f := df \circ df$ and let $g_{f,a,b} := g_{\psi_f,a,b}$. The metric $g_{f,a,b}$ is geometrically realizable as a hypersurface in $\mathbb{R}^{(u+a,u+1+b)}$ with second fundamental form given by the Hessian $H_{ij}(f) := \partial_i^x \partial_j^x f$.*

Theorem 1.5. *Let $g_{\psi,a,b}$ be the metric of Definition 1.3. Then:*

(1) *$g_{\psi,a,b}$ is Ricci flat, 2-nilpotent, and Einstein;*
(2) *$g_{\psi,a,b}$ is neither locally homogeneous nor locally symmetric for generic ψ;*
(3) *$g_{\psi,a,b}$ is Osserman, k Osserman for any admissible k, IP, and Szabó.*

We will examine these manifolds in greater detail in subsequent sections. We shall almost always assume $p \leq q$ so the spacelike directions dominate; the case $q \leq p$ can be studied by reversing the sign of the quadratic form in question.

Here is a brief guide to this paper. In Section 2, we study the Jacobi operator and in Section 3, we study the higher order Jacobi operator. We study the skew-symmetric curvature operator in the real setting in Section 4 and in the complex setting in Section 5. We conclude in Section 6 by studying the Szabó operator.

2. The Jacobi Operator

The classification of Osserman manifolds is essentially complete in the Riemannian ($p = 0$) and the Lorentzian ($p = 1$) settings. We refer to [5, 26] for the proof of assertion (1) and to [2, 6] for the proof of assertion (2) in the following Theorem:

Theorem 2.1.

(1) *Let (M, g) be a Riemannian Osserman manifold of dimension $m \neq 16$. Then (M, g) is either flat or locally isometric to a rank 1 symmetric space.*
(2) *Let (M, g) be a Lorentzian Osserman manifold. Then (M, g) has constant sectional curvature.*

If (M, g) is either flat or locally isometric to a rank 1 symmetric space, then (M, g) is both spacelike Jordan Osserman and timelike Jordan Osserman. Thus one is interested in finding examples of pseudo-Riemannian manifolds which are either spacelike Jordan Osserman or timelike Jordan Osserman, but which are neither flat nor local rank 1 symmetric spaces.

If W is an auxiliary vector space and if A is a linear map of V, then we define the *stabilization*

$$A \oplus 0 := \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \quad \text{on} \quad V \oplus W.$$

The Jordan normal form of a spacelike Jordan Osserman algebraic curvature can be arbitrarily complicated in the balanced setting [11]:

Theorem 2.2. *Let $\mathcal{J}$ be a $r \times r$ real matrix and let $p \equiv 0$ mod 2^r. If V is a vector space of neutral signature (p, p), then there exists an algebraic curvature tensor R on V so that $J(x)$ is conjugate to $\pm\mathcal{J} \oplus 0$ for every $x \in S^{\pm}(V)$.*

The situation is very different if $p \neq q$. If $p < q$, then the spacelike directions dominate in a certain sense and spacelike Jordan Osserman is a very strong condition. We refer to [15] for the proof of the following result:

Theorem 2.3. *Let R be an algebraic curvature tensor on a vector space V of signature (p, q). If $p < q$ and if R is spacelike Jordan Osserman, then $J(x)$ is diagonalizable for every $x \in S^+(V)$.*

The following result [17] deals with the opposite setting and provides examples of nilpotent timelike Jordan Osserman manifolds if $2 \leq p \leq q$:

Theorem 2.4. *Let f be a smooth function on $\mathbb{R}^u$ with positive definite Hessian. Let $g_{f,a,b}$ be the metric of signature $(u+a, u+b)$ described in Definition 1.4. Then $g_{f,a,b}$ is timelike Jordan Osserman if and only if $a = 0$.*

Following Garciá-Rió, Kupeli, and Vázquez-Lorenzo [8] (see page 147), we can describe a family which arises from affine geometry. Let $\Gamma_{ij}{}^k(x)$ be the Christoffel symbols of an arbitrary torsion free connection ∇ on $\mathbb{R}^u$. This connection is said to be *affine Osserman* if and only if the associated Jacobi operator is nilpotent. Define a metric g_∇ on $\mathbb{R}^{2u}$ by setting:

$$ds^2_\nabla = \textstyle\sum_i dx^i \circ dy^i - 2\sum_{ijk} y_k \Gamma_{ij}{}^k(x) dx^i \circ dx^j.$$

Then (M, g_∇) is Osserman if and only if ∇ is affine Osserman. This metric is quite different as the coefficients depend on the y variables as well as on the x variables; there does not seem to be any direct connection between this metric and the metrics described in Definition 1.3. We refer to [3, 4, 7] for other examples of Osserman manifolds in the higher signature setting.

3. The higher order Jacobi operator

There is a basic duality result for the higher order Jacobi operator [11, 20, 28]:

Theorem 3.1. *Let R be an algebraic curvature tensor on a vector space V of signature (p, q).*

(1) *If R is k Osserman, then R is $m - k$ Osserman.*
(2) *If R is Jordan Osserman of type (r, s), then R is Jordan Osserman of type $(p - r, q - s)$.*

As $m - 1$ Osserman is equivalent to 1 Osserman, we suppose $2 \leq k \leq m - 2$. The classification of k Osserman pseudo-Riemannian manifolds is complete in the Riemannian and Lorentzian settings [10, 21]:

Theorem 3.2. *Let (M, g) be a k Osserman pseudo-Riemannian manifold of signature (p, q), where $2 \leq k \leq m - 2$. If $p = 0$ or if $p = 1$, then (M, g) has constant sectional curvature.*

Let Ψ be the set of all symmetric 2 tensors ψ on $\mathbb{R}^u$ so that the Jacobi operator $J_\psi(v)$ defined by the metric g_ψ is positive semi-definite of rank $p - 1$ for every non zero tangent vector v in span$\{\partial_i^x\}$. For example, if f is a smooth function on $\mathbb{R}^p$ and if $H(f)$ is positive definite, then $df \circ df \in \Psi$. The set Ψ is non-empty, convex, and conelike; it is open in the measurant topology [18]. We have:

Theorem 3.3. *Let $\psi \in \Psi$ and let $g_{\psi,a,b}$ be the metric of Definition 1.3. Then:*

(1) *$g_{\psi,a,b}$ is nilpotent Jordan Osserman*
    (a) *of types $(r, 0)$ and $(p - r, q)$ if $a = 0$ and if $0 < r \leq p$;*
    (b) *of types $(0, s)$ and $(p, q - s)$ if $b = 0$ and if $0 < s \leq p$;*
    (c) *of types $(r, 0)$ and $(p - r, q)$ if $a > 0$ and if $a + 2 \leq r \leq p$;*
    (d) *of types $(0, s)$ and $(p, q - s)$ if $b > 0$ and if $b + 2 \leq s \leq q$;*
(2) *$g_{\psi,a,b}$ is not Jordan Osserman of type (r, s) otherwise.*

See also Bonome et. al. [4] for another family of higher order Osserman manifolds.

There is a final family [14] of algebraic curvature tensors which are higher order nilpotent Jordan Osserman which do not seem to be realizable geometrically. Let $2 \leq p \leq q$. Let $\{e_1^-, ..., e_p^-, e_1^+, ..., e_q^+\}$ be an orthonormal basis for V, where the vectors $\{e_1^-, ..., e_p^-\}$ are timelike and the vectors $\{e_1^+, ..., e_q^+\}$ are spacelike. Let a be a positive integer with $2a \leq p$. We define a skew-adjoint linear map Φ_a of V and associated curvature operator R_a by setting:

$$\phi_a e_k^\pm = \begin{cases} \pm(e_{2i}^- + e_{2i}^+) & \text{if} \quad k = 2i-1 \leq 2a, \\ \mp(e_{2i-1}^- + e_{2i-1}^+) & \text{if} \quad k = 2i \leq 2a, \\ 0 & \text{if} \quad k > 2a. \end{cases} \tag{3.1}$$

$$R_a(x,y)z = (y, \Phi_a z)\Phi_a x - (x, \Phi_a z)\Phi_a y - 2(x, \Phi_a y)\Phi_a z.$$

Theorem 3.4. *Let $2a \leq p \leq q$. Let R_a be the 4 tensor defined in (3.1). Then:*

(1) *R_a is an algebraic curvature tensor which is nilpotent k Osserman for any admissible k.*
(2) *If $2a < p$, then R_a is Jordan Osserman of type $(p,0)$ and $(0,q)$; R_a is not Jordan Osserman of type (r,s) otherwise.*
(3) *If $2a = p < q$, then R_a is Jordan Osserman of type $(r,0)$ and of type (r,q) for any $1 \leq r \leq p-1$; R_a is not Jordan Osserman otherwise.*
(4) *If $2a = p = q$, then R_a is Jordan Osserman of type $(r,0)$, of type (r,q), of type $(0,s)$, and of type (p,s) for $1 \leq r \leq p-1$ and $1 \leq s \leq q-1$; R_a is not Jordan Osserman otherwise.*

4. The skew-symmetric curvature operator

Bounding the rank of a spacelike Jordan IP algebraic curvature tensor is crucial. We refer to [19] for the proof of assertion (1a), to [32] for the proof of assertion (1b), to [29] for the proof of assertion (1c), and to [22] for the proof of assertions (2) and (3) in the following result.

Theorem 4.1. *Let $R \neq 0$ be an algebraic curvature tensor on a vector space of signature (p,q), where $q \geq 5$.*

(1) *If $\mathcal{R}$ has constant rank r on $Gr_{0,2}^+(V)$, then*
    (a) *If $p = 0$ and if $q = 5, 6$ or if $q \geq 9$, then $r = 2$;*
    (b) *If $p = 1$, and if $q \geq 9$, then $r = 2$;*
    (c) *If $2 \leq p \leq \frac{q-6}{4}$ and if $\{q, q+1, ..., q+p\}$ does not contain a power of 2, then $r = 2$.*
(2) *If $\mathcal{R}$ has constant rank 2 on $Gr_{0,2}^+(V)$, then there exists a self-adjoint map ϕ of V whose kernel contains no spacelike vectors so that $R = \pm R_\phi$, where*

$$R_\phi(x,y,z,w) := (\phi x, w)(\phi y, z) - (\phi x, z)(\phi y, w).$$

(3) *If $\mathcal{R}$ is spacelike Jordan IP of rank 2, then $R = \pm R_\phi$, where ϕ is self-adjoint and where $\phi^2 = C \cdot \mathrm{Id}$.*

We refer to [9] for results concerning the exceptional cases $(p,q) = (0,7)$ and $(p,q) = (0,8)$ in the Riemannian setting.

Ivanov and Petrova [24] classified the IP Riemannian manifolds of dimension $m = 4$. This classification was subsequently generalized by a number of authors. The following result follows in the Riemannian setting from [19] and in the pseudo-Riemannian setting from [23].

Theorem 4.2. *Let (M, g) be a spacelike Jordan IP pseudo-Riemannian manifold of signature (p, q), where $q \geq 5$. Assume that R_g is not nilpotent for at least one point of M and that $\mathcal{R}$ has constant rank 2. Then either (M, g) has constant sectional curvature or locally we can express $ds^2 = \varepsilon dt^2 + f(t)ds^2_\kappa$, where ds^2_κ has a metric of constant sectional curvature κ, where $f(t) = \varepsilon\kappa t^2 + At + B$, where $\varepsilon = \pm 1$, and where $A^2 - 4\varepsilon\kappa B \neq 0$.*

Theorem 4.2 focuses attention on the nilpotent spacelike Jordan Osserman manifolds since the classification is incomplete in this setting. We refer to [17] for the proof of the following result:

Theorem 4.3. *Let f be a smooth function with non-degenerate Hessian and let $g_{f,a,b}$ be the metric described in Definition 1.4. Then $g_{f,a,b}$ is:*

(1) *nilpotent IP;*
(2) *never mixed Jordan IP;*
(3) *spacelike Jordan IP if and only if $b = 0$;*
(4) *timelike Jordan IP if and only if $a = 0$.*

5. The skew-symmetric curvature operator in the complex setting

We suppose given a Hermitian almost complex structure J on V, i.e. a skew-adjoint endomorphism J of V which is an isometry such that $J^2 = -\operatorname{Id}$. We use J to give a complex structure to V by defining $\sqrt{-1}x := Jx$. A 2 plane π is said to be a *complex line* if $J\pi = \pi$; let $\mathbb{CP}(V)$ be the space of all non-degenerate complex lines. A linear transformation T is said to be *complex* if $TJ = JT$. An algebraic curvature tensor R is said to be *almost complex* if $J\mathcal{R}(\pi) = \mathcal{R}(\pi)J$ for every non-degenerate complex line π; i.e. if $\mathcal{R}(\cdot)$ is complex on $\mathbb{CP}(V)$. Equivalently, [12] R is almost complex if we have the curvature identity:

$$J^*R = R \quad \text{i.e.} \quad R(x, y, z, w) = R(Jx, Jy, Jz, Jw) \quad \forall (x, y, z, w) \in V.$$

If the Jordan normal form of such a tensor $\mathcal{R}(\cdot)$ is constant on $\mathbb{CP}(V)$, then R is said to be *almost complex Jordan IP*.

Even in the Riemannian setting, the classification is incomplete. We refer to [13] for the proof of the following result which controls the eigenvalue structure:

Theorem 5.1. *Let V have signature $(0, q)$. Let R be an almost complex Jordan IP algebraic curvature tensor on V. Let $\{\lambda_s, \mu_s\}$ be the eigenvalues and multiplicities of $J\mathcal{R}$, where we order the multiplicities μ_s so that $\mu_0 \geq \ldots \geq \mu_\ell$. Suppose that $\ell \geq 1$. If $q \equiv 2 \bmod 4$, then $\ell = 1$ and $\mu_1 = 1$. If $q \equiv 0 \bmod 4$, then either $\ell = 1$ and $\mu_1 \leq 2$ or $\ell = 2$ and $\mu_1 = \mu_2 = 1$.*

This theorem is sharp, there exist almost complex Jordan IP algebraic curvature tensors with the indicated eigenvalue structures, see [12] for details. We remark that Kath [25] has obtained some partial results concerning almost complex Jordan IP algebraic curvature tensors in signatures $(p, q) = (0, 4)$ or $(p, q) = (2, 2)$.

6. The Szabó Operator

The classification of Szabó tensors is complete in the Riemannian and Lorentzian settings; there are no non-trivial examples. We refer to Szabó [30] for the case $p = 0$ and to Gilkey-Stavrov [21] for case $p = 1$ in the following result:

Theorem 6.1. *Let $\mathfrak{R}$ be a Szabó tensor on a vector space of signature (p, q), where $p = 0$ or $p = 1$. Then $\mathfrak{R} = 0$.*

Szabó used this observation to give a proof of the well known fact that any 2 point homogeneous Riemannian manifold is locally symmetric; similarly any pointwise totally isotropic Lorentzian manifold is locally symmetric.

Theorem 1.5 provides examples of non-trivial Szabó tensors if $2 \leq p \leq q$. However, these examples are neither spacelike Jordan Szabó nor timelike Jordan Szabó if $\nabla R \neq 0$. In fact, there are no known examples of spacelike Jordan Szabó or timelike Jordan Szabó algebraic curvature tensors such that $\mathfrak{R} \neq 0$ and it is natural to conjecture that none exist. There are some partial results in this direction. Let A be self-adjoint and let $\lambda \in \mathbb{C}$. We define real operators A_λ on V and associated generalized subspaces E_λ by setting:

$$A_\lambda := \begin{cases} A - \lambda \cdot \mathrm{Id} & \text{if} \quad \lambda \in \mathbb{R}, \\ (A - \lambda \cdot \mathrm{Id})(A - \bar{\lambda} \cdot \mathrm{Id}) & \text{if} \quad \lambda \in \mathbb{C} - \mathbb{R}, \end{cases}$$

$$E_\lambda := \ker\{A_\lambda^m\}. \tag{6.1}$$

As $A_\lambda = A_{\bar{\lambda}}$, we see $E_\lambda = E_{\bar{\lambda}}$. Both A and A_λ preserve each generalized eigenspace E_λ. The operator A is said to be *Jordan simple* if $A_\lambda = 0$ on E_λ for all λ.

Let $\nu(q)$ be the *Adams number*; this is the maximal number of linearly independent vector fields on the sphere S^{q-1} in $\mathbb{R}^q$ [1]. If $\mathcal{R}$ is a Szabó tensor, then let $\mathrm{spec}^{\pm}(\mathfrak{S})$ denote the spectrum of the associated

Szabó operator on $S^{\pm}(V)$. We refer to [16] for the proof of the following theorem in the algebraic context:

Theorem 6.2. *Let $\mathfrak{R}$ be an algebraic covariant derivative curvature tensor on a vector space of signature (p, q).*

1. *If $\mathfrak{R}$ is Szabó, then:*
   - (a) $\mathrm{spec}^{\pm}\{\mathfrak{S}\} = -\mathrm{spec}^{\pm}\{\mathfrak{S}\} = \sqrt{-1}\,\mathrm{spec}^{\mp}\{\mathfrak{S}\}$*;*
   - (b) $\mathrm{spec}^{\pm}\{\mathfrak{S}\} \subset \mathbb{R} \cup \sqrt{-1}\,\mathbb{R}$*;*
   - (c) *If $p < q$, then* $\mathrm{spec}^{+}\{\mathfrak{S}\} \subset \sqrt{-1}\,\mathbb{R}$ *and* $\mathrm{spec}^{-}\{\mathfrak{S}\} \subset \mathbb{R}$.
2. *If $p < q$ and if $\mathfrak{R}$ is spacelike Jordan Szabó, then:*
   - (a) *$\mathfrak{S}(v)$ is Jordan simple for any $v \in S^+(V)$;*
   - (b) *If $p < q - \nu(q)$, then* $\mathrm{rank}\{\mathfrak{S}(v)\} \leq 2\nu(q)$ *for any $v \in S^+(V)$;*
   - (c) *If q is odd, then $\mathfrak{R} = 0$.*

We say that a pseudo-Riemannian manifold (M, g) is *pointwise totally isotropic* if given a point $P \in M$ and nonzero tangent vectors X and Y in T_PM with $(X, X) = (Y, Y)$, there is a local isometry of (M, g), fixing P, which sends X to Y. The Szabó operator of a pointwise totally isotropic pseudo-Riemannian manifold necessarily has constant Jordan normal form on $S^+(T_PM)$ for any $P \in M$. Thus Theorem 6.2 yields the following Corollary:

Corollary 6.3. *Let (M, g) be a pointwise totally isotropic pseudo-Riemannian manifold of signature (p, q), where $p < q$ and where q is odd. Then (M, g) is locally symmetric.*

More generally, Wolf showed that any locally totally isotropic manifold is necessarily locally symmetric, see [31] (Theorem 12.3.1). Thus Corollary 6.3 gives a different proof of Wolf's result if $p < q$ and if q is odd.

References

[1] J. Adams, *Vector fields on spheres*, Annals of Math., **75**, (1962), 603–632.

[2] N. Blažic, N. Bokan, and P. Gilkey, A note on Osserman Lorentzian manifolds, *Bulletin of the London Math Society*, **29**, (1997), 227–230.

[3] N. Blažić, N. Bokan, P. Gilkey and Z. Rakić, Pseudo-Riemannian Osserman manifolds, *J. Balkan Soc. of Geometers*, **l2**, (1997), 1–12.

[4] A. Bonome, P. Castro, E. Garcia-Rio, Four-Dimensional Generalized Osserman Manifolds, *Classical and Quantum Gravity*, **18**, (2001), 4813–4822.

[5] Q.-S. Chi, A curvature characterization of certain locally rank-one symmetric spaces, *J. Differ. Geom.*, **28**, (1988), 187–202.

[6] E. García-Rio, D. Kupeli, and M. Vázquez-Abal, On a problem of Osserman in Lorentzian geometry, *Diff. Geom. Appl.*, **7**, (1997), 85–100.

[7] E. García-Rió, M. E. Vázquez-Abal and R. Vázquez-Lorenzo, Nonsymmetric Osserman pseudo-Riemannian manifolds, *Proc. Amer. Math. Soc.*, **126**, (1998), 2771–2778.

[8] E. Garciá-Rió, D. Kupeli, and R. Vázquez-Lorenzo, **Osserman Manifolds in Semi-Riemannian Geometry**, Lecture notes in Mathematics, Springer Verlag, (2002), ISBN 3-540-43144-6.

[9] P. Gilkey, Riemannian manifolds whose skew-symmetric curvature operator has constant eigenvalues II, **Differential geometry and applications** (eds Kolar, Kowalski, Krupka, and Slovák) Publ. Masaryk University, Brno, Czech Republic, (1999), ISBN 80-210-2097-0, 73–87.

[10] —, Algebraic curvature tensors which are p Osserman, *Diff. Geometry and Appl.*, **14**, (2001), 297–311.

[11] —, **Geometric Properties of Natural Operators Defined by the Riemann Curvature Tensor**, World Scientific Press, ISBN 981-02-04752-4, (2001).

[12] P. Gilkey and R. Ivanova, The geometry of the skew-symmetric curvature operator in the complex setting, **Global Differential Geometry: The Mathematical Legacy of Alfred Gray**, Ed. M Fernandez and J. Wolf, Contemporary Mathematics, **288**, (2001), ISBN 0-8218-2750-2, 325-333.

[13] —, Bundles over projective spaces and algebraic curvature tensors, *J. Geom.,* **71**, (2001), 54–67.

[14] —, The Jordan normal form of higher order Osserman algebraic curvature tensors, *Comment. Math. Univ. Carolinae*, **43**, (2002), 231–242.

[15] —, Spacelike Jordan Osserman algebraic curvature tensors in the higher signature setting, to appear in the Proceedings of the Valencia Conference on Differential Geometry, (2001), preprint http://arXiv.org/abs/math.DG/0205072.

[16] —, Spacelike Jordan Szabó algebraic covariant curvature tensors in the higher signature setting, preprint.

[17] P. Gilkey, R. Ivanova, and T. Zhang, Szabó Osserman IP pseudo-Riemannian manifolds, preprint http://arXiv.org/abs/math.DG/0205085.

[18] —, Higher order Jordan Osserman pseudo-Riemannian manifolds, preprint http://arXiv.org/abs/math.DG/0205269.

[19] P. Gilkey, J. V. Leahy and H. Sadofsky, Riemannian manifolds whose skew-symmetric curvature operator has constant eigenvalues, *Indiana Univ. Math. J.*, **48**, (1999), 615–634.

[20] P. Gilkey, G. Stanilov and V. Videv, Pseudo-Riemannian manifolds whose generalized Jacobi operator has constant characteristic polynomial, *J. Geom.*, **62**, (1998), 144–153.

[21] P. Gilkey and I. Stavrov, Curvature tensors whose Jacobi or Szabó operator is nilpotent on null vectors, *Bull. London Math. Soc.*, to appear; preprint http://arXiv.org/abs/math.DG/0205074

[22] P. Gilkey and T. Zhang, Algebraic curvature tensors whose skew-symmetric curvature operator has constant rank 2, *Periodica Mathematica Hungarica*, **44**, (2002), 7–26.

[23] —, Algebraic curvature tensors for indefinite metrics whose skew-symmetric curvature tensor has constant Jordan normal form, *Houston Math. J.*, **28**, (2002), 311–328.

[24] S. Ivanov and I. Petrova, Riemannian manifolds in which the skew-symmetric curvature operator has pointwise constant eigenvalues, *Geom. Dedicata*, **70**, (1998), 269–282.

[25] I. Kath, Almost complex algebraic curvature tensors in dimension 4, preprint.

[26] Y. Nikolayevsky, Osserman Conjecture in dimension $n \neq 16$, preprint http://arXiv.org/abs/math.DG/0204258.
[27] R. Osserman, Curvature in the eighties, *Amer. Math. Monthly*, **97**, (1990), 731–756.
[28] G. Stanilov and V. Videv, Four dimensional pointwise Osserman manifolds, *Abh. Math. Sem. Univ. Hamburg*, **68**, (1998), 1–6.
[29] I. Stavrov, Ph. D. Thesis, University of Oregon (2003).
[30] Z. I. Szabó, A short topological proof for the symmetry of 2 point homogeneous spaces, *Invent. Math.*, **106**, (1991), 61–64.
[31] J. Wolf, **Spaces of constant curvature,** 5th ed., Publish or Perish Inc., ISBN 0-914098-07-1, (1984).
[32] T. Zhang, Applications of algebraic topology in bounding the rank of the skew-symmetric curvature operator, *Topology and Appl.*, (to appear).

PG: Mathematics Department, University of Oregon, Eugene OR 97403 USA.
email: gilkey@darkwing.uoregon.edu

RI: Mathmatics Department, University of Hawaii - Hilo, 200 W. Kawili St., Hilo HI 96720 USA
email: rivanova@hawaii.edu

TZ: Department of Mathematics and Statistics, Murray State University, Murray KY 42071 USA
email: tan.zhang@murraystate.edu