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SECTIONS OF RIEMANNIAN FIBRATION

JIN-WHAN YIM

Abstract. In Riemannian geometry many geometric objects are described as sections of fibre bundles. A section of a Riemannian fibration is called harmonic if it is harmonic as a map from the base manifold into the total space. When the fibres are totally geodesic, the Euler-Lagrange equation for such sections is formulated. In the case of vector fields or distributions, which are sections of a tangent bundle or a Grassmannian bundle, this formula is described in terms of the geometry of base manifolds. Examples of harmonic distributions are considered when the base manifolds are homogeneous spaces and the integral submanifolds are totally geodesic.

1. Introduction

In Riemannian geometry many geometric objects are described as sections of fibre bundles, and their geometric properties are characterized by the bundle structure. In this article, we deal with some special kind of sections of Riemannian fibration.

Let M and N be complete Riemannian manifolds. Assume M is compact. A smooth map $f : M \to N$ is called harmonic if it is a critical point of the energy functional. Consider the case when N is a fibre bundle over M and $f : M \to N$ is a smooth map which happens to be a section of this fibration. For example, a vector field is a section of a tangent bundle, and in general a tensor field is a section of a tensor bundle over a Riemannian manifold. If N is a Grassmannian bundle associated to the orthogonal bundle $O(M)$, then the map f defines a distribution on M. We consider a connection type metric on the bundle and a section will be called harmonic if it is harmonic as a map into the total space. We will find the Euler-Lagrange equation for f in a general setting and apply this to find examples of harmonic distributions.

The definition of harmonic sections in general, is used by other authors [8], [9], [10], [14], in a similar context but for a different concept. They call a section harmonic if its vertical energy is stationary with respect to vertical variations. If one is looking for a better section, this notion of harmonicity makes more sense because a variation through sections would be a vertical one and the horizontal energy does not change.

In [7], it was proved that a unit vector field on S^3 is a harmonic map into its unit tangent bundle if and only if it is tangent to the Hopf-fibration. This result

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was improved by Gluck and Gu [6] that a unit vector field is tangent to the Hopf-fibration if it is stationary for horizontal variations. They also gave an interesting interpretation of such horizontally harmonic vector fields on the standard 3-sphere that it corresponds to the so-called Beltrami field in fluid mechanics. This horizontal harmonicity apparently carries more geometry of manifold, but its geometric meaning is not yet well understood.

In the following sections, we will consider sections of a fibre bundle with totally geodesic fibres, and find a formula of tension fields for sections as maps into the total space. In the case of Grassmannian bundles, this tension field will be written in terms of geometry of the base manifold. In the last section we will provide examples of harmonic distributions on Riemannian manifolds. The only known example on which the horizontal distribution is completely understood is the round sphere S^3 we mentioned above. When a vector field is harmonic on S^3, it is invariant under $SU(2)$-action and its integral curves are necessarily geodesics. Since it is extremely difficult to find a harmonic map for general Riemannian manifolds, we follow this line of reasoning, and study invariant distributions on homogeneous spaces with totally geodesic integral submanifolds. There is a standard way of constructing such manifolds. In fact, for compact Lie groups $K \subset H \subset G$ with suitable metric, the natural fibration

$$\pi : M = G/K \to G/H$$

has the totally geodesic fibre H/K. It is known that some homogeneous Einstein spaces can be constructed in this way, and in particular this construction includes distributions tangent to fibres in all of the generalized Hopf-fibrations. We will be able to show that all the generalized Hopf-fibrations define harmonic distributions in our sense.

The content of this article is basically a summary of results in [2] and [7], to which we will refer for details and proofs. We also refer to [1], [3], [4] for basic tools and more detailed description of harmonic maps between Riemannian manifolds.

2. Riemannian fibration and harmonicity

In this section, we will describe the Euler-Lagrange equation for the harmonic map, which is a section of a Riemannian fibration with totally geodesic fibres.

Let M and N be complete Riemannian manifolds. Assume M is compact. A smooth map $\pi : N \to M$ is called a Riemannian submersion if π is a submersion and if for each $x \in N$, the horizontal subspace of T_xN (orthogonal to the fibre over $\pi(x)$ in N) is mapped isometrically by $d\pi|_x$ to $T_{\pi(x)}M$. We denote by $\mathcal{H}$, $\mathcal{V}$ the horizontal and the vertical distribution, respectively. Then we can decompose the tangent bundle $TN = TN^{\mathcal{H}} \oplus TN^{\mathcal{V}}$, where we denote by $TN^{\mathcal{H}}$, $TN^{\mathcal{V}}$ the horizontal and the vertical subbundle, respectively.

We now consider a Riemannian submersion with totally geodesic fibre F, that is, for each x in N with $p = \pi(x)$, $\pi^{-1}(p) = F_x$ is a totally geodesic submanifold of N. Then all the fibres are isometric to each other and π is a Riemannian fibration [1]. Furthermore, the horizontal distribution defines a connection on this fibre bundle. Let $f : M \to N$ be a smooth map which happens to be a section. The section f is a harmonic map if and only if it is a critical point of the energy functional $E(f) = \int_M e(f)dv$, where $e(f) = \frac{1}{2}\|df\|^2$ is the energy density of f. The differential map df is a differential 1-form with values in the pull-back bundle $f^{-1}(TN)$ and hence a section of $T^*M \otimes f^{-1}(TN)$. Decompose $f^{-1}(TN)$ as $f^{-1}(TN^{\mathcal{H}}) \oplus f^{-1}(TN^{\mathcal{V}})$, and then we have $df = df^{\mathcal{H}} + df^{\mathcal{V}}$, where $df^{\mathcal{H}} \in \Gamma\big(T^*M \otimes f^{-1}(TN^{\mathcal{H}})\big)$, $df^{\mathcal{V}} \in \Gamma\big(T^*M \otimes f^{-1}(TN^{\mathcal{V}})\big)$. Then the energy $E(f)$ is given by

$$E(f) = E^{\mathcal{H}}(f) + E^{\mathcal{V}}(f) = \frac{1}{2}\int_M \|df^{\mathcal{H}}\|^2 dv + \frac{1}{2}\int_M \|df^{\mathcal{V}}\|^2 dv.$$

Since f is a section of a Riemannian fibration, the linear map $df_p^{\mathcal{H}} : T_pM \to (T_xN)^{\mathcal{H}}$ is an isometry for each $p = \pi(x)$, and hence we have $E^{\mathcal{H}}(f) = \frac{m}{2}\,\mathrm{Vol}(M)$ (dim $M = m$).

In the case of a vector bundle with metric connection, it is easy to see that a section is harmonic if and only if it is parallel. In fact, for a section $f : M \to N$, consider a variation of f given by $f_t(p) = tf(p), p \in M$. For an orthonormal basis $\{e_i\}$ of T_pM, $df(e_i) = \tilde{e}_i + \nabla_{e_i} f$, where $\tilde{e}_i$ is the horizontal lift of e_i. Hence $df_t(e_i) = \tilde{e}_i + t\nabla_{e_i} f$, and

$$E(f_t) = \frac{1}{2}\int_M \|df_t\|^2 dv = \frac{1}{2}\Big(m\mathrm{Vol}(M) + t^2\int_M \|\nabla_{e_i} f\|^2 dv\Big).$$

Therefore $\frac{d}{dt}\Big|_{t=1} E(f_t) = 0$ implies $\nabla f \equiv 0$ (In the case of tangent bundle, see [10]).

For $f : M \to N$ we now consider the Euler-Lagrange equation of the energy functional. Let ∇, $\tilde{\nabla}$ be the Levi-Civita connection on M and N, respectively, and let $\bar{\nabla}$ be the induced connection on the pull-back bundle. Then we have $\bar{\nabla}(df) \in \Gamma((S^2M \otimes f^{-1}(TN^{\mathcal{H}})) \oplus (S^2M \otimes f^{-1}(TN^{\mathcal{V}})))$, where S^2M is the space of symmetric covariant 2-tensors. Taking trace of the second fundamental form gives the tension field,

$$\tau(f) = -\bar{\nabla}^*(df) = Tr(\bar{\nabla}df) \in \Gamma(f^{-1}(TN)).$$

Then $f : M \to N$ is a harmonic map if and only if $\tau(f) = 0$. In fact, for a vector field V along f, let $\Sigma : (-\epsilon, \epsilon) \times M \to N$ be a variation such that $\Sigma(0, p) = f(p)$, $\frac{\partial \Sigma}{\partial t}(0, p) = V(p)$, and $f_t(p) := \Sigma(t, p)$. Then

$$\frac{d}{dt}\Big|_{t=0} E(f_t) = -\int_M \langle V, \tau\rangle_N dv,$$

where $\langle\cdot,\cdot\rangle_N$ denotes the Riemannian metric on N. We decompose τ as $\tau = \tau^{\mathcal{H}} + \tau^{\mathcal{V}}$, $\tau^{\mathcal{H}} \in \Gamma(f^{-1}(TN^{\mathcal{H}}))$ and $\tau^{\mathcal{V}} \in \Gamma(f^{-1}(TN^{\mathcal{V}}))$. Then we have

$$\tau^{\mathcal{H}} = (Tr\bar{\nabla}df^{\mathcal{H}})^{\mathcal{H}} + (Tr\bar{\nabla}df^{\mathcal{V}})^{\mathcal{H}},$$
$$\tau^{\mathcal{V}} = (Tr\bar{\nabla}df^{\mathcal{H}})^{\mathcal{V}} + (Tr\bar{\nabla}df^{\mathcal{V}})^{\mathcal{V}}.$$

Denote by ${}^{\mathcal{H}}\bar{\nabla}$, ${}^{\mathcal{V}}\bar{\nabla}$ the horizontal and the vertical component of $\bar{\nabla}$, respectively. Since $Tr({}^{\mathcal{H}}\bar{\nabla}df^{\mathcal{H}}) = (Tr\bar{\nabla}df^{\mathcal{H}})^{\mathcal{H}}$ and $Tr({}^{\mathcal{V}}\bar{\nabla}df^{\mathcal{V}}) = (Tr\bar{\nabla}df^{\mathcal{V}})^{\mathcal{V}}$, we use the rough Laplacian notation of induced connections.

Definition 1. *For $f : M \to N$ as above, we denote*

$$\Delta^{\mathcal{H}} = {}^{\mathcal{H}}\bar{\nabla}^*(df^{\mathcal{H}}) = -Tr({}^{\mathcal{H}}\bar{\nabla}df^{\mathcal{H}}),$$
$$\Delta^{\mathcal{V}} = {}^{\mathcal{V}}\bar{\nabla}^*(df^{\mathcal{V}}) = -Tr({}^{\mathcal{V}}\bar{\nabla}df^{\mathcal{V}}).$$

Then we have the following theorem. (See [2].)

Theorem 1. *For $f : M \to N$ as above, we have*

$$\tau(f) = \tau^{\mathcal{H}} + \tau^{\mathcal{V}} = -\big(2\Delta^{\mathcal{H}} + \Delta^{\mathcal{V}}\big).$$

Let $\bar{\pi} : P \to M$ be the principal G-bundle associated to N, F fibre space on which G acts, and $\varphi : P \times F \to N$ the principal map. Let ω and Ω be the corresponding connection form and the curvature form on P. For $p \in M$, choose $(u, \xi) \in P \times F$ such that $\varphi(u, \xi) = f(p)$. We then fix ξ and consider the map $\varphi_\xi : P \to N$, $\varphi_\xi(\alpha) = \varphi(\alpha, \xi)$. The connection on N is associated to that of P, and therefore the horizontal space of N is, by definition, the image of the horizontal space of P by φ_ξ. Let $\bar{e}_i$ be the horizontal vector field on P such that $d\varphi_\xi(\bar{e}_i) = \tilde{e}_i$. Then by the structure equation we have $\omega_u([\bar{e}_i, \bar{e}_j]) = -\Omega_u(\bar{e}_i, \bar{e}_j)$, and hence

$$[\tilde{e}_i, \tilde{e}_j]^{\mathcal{V}} = d\varphi_\xi([\bar{e}_i, \bar{e}_j]^{\mathcal{V}}) = -d\varphi_\xi\Big((\omega_u|_{\mathcal{V}})^{-1}(\Omega_u(\bar{e}_i, \bar{e}_j))\Big).$$

With this observation, it is not difficult to prove the following.

Proposition 1. *Let $\pi : N \to M$ be a Riemannian submersion with totally geodesic fibres and $f : M \to N$ a section. Then*

$$\tau^{\mathcal{H}} = \sum_{i,j} \big\langle d\varphi_\xi\big((\omega_u|_{\mathcal{V}})^{-1}(\Omega_u(\bar{e}_i, \bar{e}_j))\big), df^{\mathcal{V}}(e_i)\big\rangle_N \tilde{e}_j,$$

where $\{\tilde{e}_i\}$, $\{\bar{e}_i\}$ are the horizontal lifts to P and N of an orthonormal frame field $\{e_i\}$ of M.

We can apply the above results to smooth distributions on Riemannian manifolds. We note that in general it is not very difficult to find an equation for harmonic maps between two Riemannian manifolds as long as Riemannian metrics are given, but we need the equation in a specific form so that we can find solutions.

A k-dimensional distribution on an n-dimensional manifold M is a smooth section of the Grassmannian bundle $G_k(M)$ of k-dimensional planes in tangent spaces of M. Throughout this section we use the index convention,

$$1 \le i, j \le k, \ \ k+1 \le \alpha, \beta \le n, \ \ 1 \le A, B \le n.$$

The Grassmannian bundle $G_k(M)$ is associated to the orthogonal bundle $O(M)$, which is a principal $O(n)$-bundle, and has the fibre $G_k(\mathbb{R}^n) = O(n)/(O(k) \times O(n-k))(= G/K)$. An invariant metric on $G_k(\mathbb{R}^n)$ is determined by an $\mathrm{Ad}_G(K)$-invariant inner product on a subspace $\mathfrak{m} \subset \mathfrak{o}(n)$, which is an $\mathrm{Ad}_G(K)$-invariant complement to $(\mathfrak{o}(k) \times \mathfrak{o}(n-k))$ and is identified with $T_{(eK)}G_k(\mathbb{R}^n)$ by the projection map. Let E^A_B denote the $(n \times n)$-matrix such that (A, B)-th entry is 1 and others are all zero. Then for the standard metric on $G_k(\mathbb{R}^n)$, we take $\mathfrak{m}(\mathbb{R}^n)$ as a subspace with the orthonormal basis $\{e^i_\alpha := E^\alpha_i - E^i_\alpha\}$.

Consider the principal map $\varphi : O(M) \times G_k(\mathbb{R}^n) \to G_k(M)$. Let ξ be the origin of $G_k(\mathbb{R}^n)$, i.e., the coset $O(k) \times O(n-k)$. Then $\varphi_\xi : O(M) \to G_k(M)$ is nothing but the projection $O(T_pM) \to G_k(T_pM)$ on fibre over each $p \in M$, and $d\varphi_\xi : \mathfrak{o}(T_pM) \to \mathfrak{m}(T_pM)$ is the corresponding projection in Lie algebra.

Let $\mathcal{D}$ be a k-dimensional distribution on M, which we also denote by a map $f : M \to G_k(M)$. Denote by $\mathcal{D}^\perp$ the orthogonal complement to $\mathcal{D}$. Choose a local orthonormal frame field $\{e_A\}$ on an open subset U of M such that

$$e_i \in \mathcal{D}, \ 1 \le i \le k, \ \ e_\alpha \in \mathcal{D}^\perp, \ k+1 \le \alpha \le n.$$

Then this frame field defines a local section $\sigma : U \to O(U)$ such that $\varphi_\xi \circ \sigma = f|_U$. With respect to this frame field, we can locally trivialize $O(U)$ as $U \times O(n)$, and through φ_ξ we obtain a local trivialization $U \times G_k(\mathbb{R}^n)$ of $G_k(U)$. Then the distribution $\mathcal{D}$ appears as the origin in $G_k(\mathbb{R}^n)$. Let ω and Ω be the connection form and curvature form of this principal bundle. For $p \in M$ let $u \in O(M)$ denote the orthonormal basis $\{e_A\}$ of T_pM. Then for $v \in T_pM$, $d\sigma^{\mathcal{V}}(v) = (\omega_u|_{\mathcal{V}})^{-1}(\omega_u(d\sigma(v))$ is a skew-symmetric matrix whose (A, B)-th entry is given by $\langle \nabla_v e_B, e_A\rangle$, where $\langle\cdot,\cdot\rangle$ is the metric of M. Note that with respect to our trivialization, $(\omega_u|_{\mathcal{V}})$ is simply the identification of a tangent space of $O(n)$ to its Lie algebra. We now improve Proposition 1 to have a more explicit formula for the horizontal tension field of a distribution. We denote by $(\cdot)^\top$ and $(\cdot)^\perp$ the projection onto $\mathcal{D}$ and $\mathcal{D}^\perp$, respectively. We then have the following description of tension field of distribution.

Theorem 2. *For a distribution $\mathcal{D}$ on M, let $\{e_A\}$ be a local orthonormal frame field such that $e_i \in \mathcal{D}$. We identify $\tau^{\mathcal{H}}(\mathcal{D})$ with its image on M by the isometry $df^{\mathcal{H}}$: $T_pM \to T_{f(p)}N^{\mathcal{H}}$. Furthermore, $\mathfrak{m}(T_pM)$ is identified as a space of skew-symmetric operators on T_pM, which can be decomposed as $Hom(\mathcal{D}, \mathcal{D}^\perp) \oplus Hom(\mathcal{D}^\perp, \mathcal{D})$, and $\tau^{\mathcal{V}}(\mathcal{D}) \in \mathfrak{m}(T_pM)$ is regarded as an element in $Hom(\mathcal{D}, \mathcal{D}^\perp)$. Then*

$$\tau^{\mathcal{H}}(\mathcal{D}) = \sum_{i,A} R((\nabla_{e_A} e_i)^\perp, e_i)e_A,$$

where R is the curvature tensor of M, and,

$$\tau^{\mathcal{V}}(\mathcal{D})(v) = \sum_A \Big((\nabla_{e_A}(\nabla_{e_A} v)^\perp)^\perp - (\nabla_{e_A}(\nabla_{e_A} v)^\top)^\perp - (\nabla_{\nabla_{e_A} e_A} v)^\perp\Big).$$

As an example, we consider the case when $M = S^n$ is a round sphere with constant curvature $K > 0$ and $\mathcal{D}$ is a unit vector field ξ . Then the curvature tensor R is given by $R(X, Y)Z = K(\langle X, Z\rangle Y - \langle Y, Z\rangle X)$. Choose an orthonormal frame $\{e_A\}$, $i = 1, \cdots, n$. Since $\langle \nabla_v \xi, \xi\rangle = 0$ for any vector v, the horizontal tension field of ξ is

$$\begin{aligned}
\tau^{\mathcal{H}}(\xi) &= \sum_{A=1}^{n} R((\nabla_{e_A}\xi)^\perp, \xi)e_A \\
&= K \sum_{A=1}^{n} (\langle \nabla_{e_A}\xi, e_A\rangle \xi - \langle \xi, e_A\rangle \nabla_{e_A}\xi) \\
&= K\big((\operatorname{div}\xi)\xi - \nabla_\xi \xi\big),
\end{aligned}$$

where $\operatorname{div}\xi$ denotes the divergence of the vector field ξ. Therefore ξ is horizontally harmonic if and only if $\operatorname{div}\xi = \nabla_\xi\xi = 0$, which means the flow is volume preserving and the integral curves are geodesics.

3. Examples of harmonic sections

In this section, we will find examples of Riemannian manifolds with harmonic sections.

We first consider a unit tangent vector field on a sphere S^{2n+1}. The unit tangent bundle US^{2n+1} is a Riemannian manifold with the Sasaki metric, which is the same metric we discussed in the previous section. Then it is easy to see that the Hopf vector fields on S^{2n+1} are harmonic maps into the unit tangent bundle. In fact, the horizontal tension field vanishes because the Hopf vector field is volume preserving and its integral curves are geodesics. The vertical component can be computed using a standard basis.

Along each fibre of the Hopf fibration we choose a standard basis $\{e_i\}, i = 0, \cdots, 2n$, such that e_0 is tangent to the fiber and $\{e_1, \cdots, e_{2n}\}$ is the horizontal lift of the normal unitary basis $\{f_i\}$ of $\mathbb{CP}^n$, i.e., $\nabla_{f_i} f_j = 0$ and $Jfe_{2i} = f_{2i+1}$ for the complex structure J. Then along the fibre $\nabla_{e_i} e_j$, $i, j \geq 1$, is a vertical vector, and in fact $\{e_i\}$ may be chosen so that $\nabla_{e_{2i}} e_{2i+1} = e_0 = -\nabla_{e_{2i+1}} e_{2i}$ and all others vanish. We further have the following table of covariant derivatives;

$$\nabla_{e_0} e_0 = 0, \quad \nabla_{e_0} e_{2i} = -e_{2i+1} = \nabla_{e_{2i}} e_0, \quad \nabla_{e_0} e_{2i+1} = e_{2i} = \nabla_{e_{2i+1}} e_0,$$

and all other covariant derivatives vanish. Using these formulae of covariant derivatives, we can now calculate the vertical tension field of ξ.

$$\begin{aligned} \tau(\xi)^V = & \ \mathrm{trace} \nabla^2 \xi \\ = & \ \textstyle\sum_{i=0}^{2n} (\nabla_{e_i} \nabla_{e_i} \xi - \nabla_{\nabla_{e_i} e_i} \xi) \\ = & \ \textstyle\sum_{i=1}^{n} (\nabla_{e_{2i}} \nabla_{e_{2i}} e_0 + \nabla_{e_{2i-1}} \nabla_{e_{2i-1}} e_0) \\ = & \ \textstyle\sum_{i=1}^{n} (-\nabla_{e_{2i}} e_{2i+1} + \nabla_{e_{2i-1}} e_{2i}) \\ = & \ -2n e_0. \end{aligned}$$

Since this vector is perpendicular to US^{2n+1}, we conclude that the Hopf vector fields are harmonic in US^{2n+1}.

In [7], it is shown that on the 3-dimensional unit sphere, the converse is also true and a smooth unit vector field is a harmonic map into the unit tangent bundle with the Sasaki metric if and only if it is the tangent vector field of the Hopf-fibration. A crucial fact about the Hopf vector field is that it is invariant under $SU(n+1)$ action on $S^{2n+1} = SU(n+1)/SU(n)$ and the integral curves are geodesics. Based on this observation, we find more general examples of symmetric spaces with harmonic distributions. Here we note that in the sphere case the metric on $SU(n+1)$ is not the standard bi-invariant metric. In general, an irreducible symmetric space G/K presented by a symmetric pair (G, K) does not carry any G-invariant distribution, and it is not as easy as one may expect to carry out computations on these symmetric spaces.

Let $M = G/K$ be a reductive homogeneous space with transitive action by a Lie group G. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k} \subset \mathfrak{g}$ the corresponding subalgebra for K, and choose an $\mathrm{Ad}_G(K)$-invariant complement $\mathfrak{m}$ to $\mathfrak{k}$ in $\mathfrak{g}$. A distribution $\mathcal{D}$ is called G-invariant if $d\gamma(\mathcal{D}_p) = \mathcal{D}_{\gamma(p)}$ for each $p \in M$ and $\gamma \in G$. It is then easy to see that by the isomorphism $\mathfrak{m} \to T_{(eK)}M$ there is a one-to-one correspondence between $\mathrm{Ad}_G(K)$-invariant subspaces of $\mathfrak{m}$ and G-invariant distributions on M.

In order to produce examples with harmonic distributions, we now recall a standard technique to construct homogeneous Riemannian fibration with totally geodesic fibres (see [1]).

Let G be a compact Lie group, H, K two compact subgroups of G with $K \subset H$. Then we have the natural fibration

$$\pi : M = G/K \to G/H$$

with fibre H/K. We further assume that G/H is an irreducible symmetric space with the symmetric pair (G, H). Let $\mathfrak{k} \subset \mathfrak{h} \subset \mathfrak{g}$ be the Lie algebras of K, H, and G, respectively. We choose an $\mathrm{Ad}_G(H)$-invariant complement $\mathfrak{n}$ to $\mathfrak{h}$ in $\mathfrak{g}$, and an $\mathrm{Ad}_G(K)$-invariant complement $\mathfrak{p}$ to $\mathfrak{k}$ in $\mathfrak{h}$. Then $\mathfrak{m} := \mathfrak{p} \oplus \mathfrak{n}$ is an $\mathrm{Ad}_G(K)$-invariant complement to $\mathfrak{k}$ in $\mathfrak{g}$, and we have $[\mathfrak{h}, \mathfrak{n}] \subset \mathfrak{n}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. Furthermore, since (G, H) is a symmetric pair we have $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{h}$. An $\mathrm{Ad}_G(H)$-invariant scalar product on $\mathfrak{n}$ defines a G-invariant Riemannian metric $\check{g}$ on G/H, and an $\mathrm{Ad}_G(K)$-invariant scalar product on $\mathfrak{p}$ defines a H-invariant Riemannian metric $\hat{g}$ on H/K. Finally, the orthogonal direct sum for these scalar products on $\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{n}$ defines a G-invariant Riemannian metric g on G/K. Then we see that the map π is a Riemannian submersion from $(G/K, g)$ to $(G/H, \check{g})$ with totally geodesic fibres isometric to $(H/K, \hat{g})$.

For example, let $G = SU(n+1), H = S(U(1)\times U(n))$ and $K = SU(n)$. Then we see that the corresponding fibration is $\pi : S^{2n+1} \to \mathbb{C}P^n$, which is called the Hopf-fibration. Notice that the standard $SU(n+1)$-invariant metric g on S^{2n+1} is *not* the round metric. However by re-scaling the metric g along the fibre we can produce the standard round sphere, and obtain an $SU(n+1)$-invariant distribution $\mathcal{D}$ tangent to the Hopf-fibration. In general, for the homogeneous spaces $M = G/K$ as above, we need to change the bi-invariant metric on G in order to obtain interesting metrics on M. We show that this type of modification can be done in a general setting.

Let $\langle\langle\cdot,\cdot\rangle\rangle$ be a bi-invariant metric on G, and let $\langle\cdot,\cdot\rangle$ a new left invariant metric defined by re-scaling $\langle\langle\cdot,\cdot\rangle\rangle$ on $\mathfrak{h}$ such that $\langle\cdot,\cdot\rangle|_{\mathfrak{h}} = \lambda\langle\langle\cdot,\cdot\rangle\rangle$ for a constant $\lambda > 0$. It is easy to see that this new metric on G is still H-biinvariant, and hence its restriction on $\mathfrak{m}$ is $\mathrm{Ad}_G(K)$-invariant. Therefore, it determines an invariant metric on $M = G/K$ that we also denote by $\langle\cdot,\cdot\rangle$. The projection $\pi : G \to M$ is a Riemannian submersion. By choosing $\lambda > 0$, we produce a family of new metrics on M, and it is known that some homogeneous Einstein manifolds can be produced in this way [1]. In particular, this construction includes all of the generalized Hopf-fibrations. Since this new metric on G is not biinvariant and M is not necessarily a normal homogeneous space, it takes some effort to understand the geometry of M in terms of the algebraic structures.

Let $M = G/K$ be a homogeneous space with a metric $\langle\cdot,\cdot\rangle$ defined as above. The compact Lie group G acts on M by isometries and hence also acts on the Grassmannian bundle $G_k(M)$, $k = \dim(\mathfrak{p})$, with the standard metric as in Section 3. The $\mathrm{Ad}_G(K)$-invariant space $\mathfrak{p}$ generates a G-invariant distribution on M, which we denote by $\mathcal{D}$, and hence $\mathcal{D}^\perp$ is generated by $\mathfrak{n}$. Since $\mathcal{D}$ is G-invariant, so is the tension vector field $\tau(\mathcal{D})$ on $G_k(M)$. Since the G-action on $G_k(M)$ clearly preserves the horizontal and vertical spaces, we see that $\tau^{\mathcal{V}}(\mathcal{D})$ and $\tau^{\mathcal{H}}(\mathcal{D})$ are both G-invariant. Moreover, $\tau^{\mathcal{H}}(\mathcal{D})$ can be identified with its image on M by the isometry $TG_k(M)^{\mathcal{H}} \to TM$, and hence it is a G-invariant vector field on M. Therefore there exists a corresponding $\mathrm{Ad}_G(K)$-invariant element in $\mathfrak{m}$, which we again denote by $\tau^{\mathcal{H}}(\mathcal{D})$. We will not distinguish these vector fields because no confusion will be caused. Then we have the following(cf. [2]).

Theorem 3. *For $M = G/K$ with a metric modified by any $\lambda > 0$, this distribution $\mathcal{D}$ defines a harmonic map from M to $G_k(M)$.*

As we mentioned, all of the generalized Hopf-fibrations are constructed in this way, and hence distributions tangent to the Hopf fibre are harmonic maps into the Grassmannian bundle.

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Dept. of Mathematics, KAIST
E-mail address: `jwyim@math.kaist.ac.kr`