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A NOTE ON CRITICAL POINT EQUATIONS

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Abstract. On a compact $n-$dimensional manifold M^n, a critical point of the total scalar curvature functional, restricted to the space of metrics with constant scalar curvature of volume 1, satifies the critical point equation (CPE), given by $z_g = s_g^*(f)$. It has been conjectured that a solution (g, f) of CPE is Einstein. Restricting our considerations to $n = 3$ and assuming that $Ker(s_g'^*) \neq 0$ throughout the present paper, we surveyed the recent developments of this conjecture.

1. Introduction

Let M be a $n-$dimensional compact manifold and $\mathcal{M}$ be the space of C^∞ Riemannian metrics on M. Then for any $g \in \mathcal{M}$, its scalar curvature s_g is an element in the space $C^\infty(M)$ functions. The derivative s_g' at $g \in \mathcal{M}$ is given by

$$s_g'(h) = \frac{d}{dt}|_{t=0} s_{g+th} = -\Delta_g trh + \delta_g^* \delta_g h - g(h, Ric)$$

Using Stoke's theorem, the L_2-adjoint operator $s_g'^*$ of s_g' with respect to the canonical L_2-inner product defined by g is computed to be

$$s_g'^*(f) = -g\Delta_g f + D_g df - f r_g$$

Regarding s_g' as a linear map from the space of symmetric tensors to the space of functions, it is known that the following decomposition are true [1]:

$$C^\infty(M) = Im(s_g') \oplus Ker(s_g'^*)$$

If $Ker(s_g'^*) \neq 0$, it is known that s_g is a nonnegative constant. More precisely, we have

Theorem 1. [3], [4] *If $Ker\ s_g'^* \neq 0$, either (M, g) is Ricci-flat and $Ker\ s_g'^* = \mathbb{R} \cdot 1$, or s_g is a positive constant and $Ker\ s_g'^* \subset Ker\ (\Delta_g - \frac{s_g}{n-1})$.*

The converse of Theorem 1 does not holds in general; for a standard product metric g_0 on $S^2 \times S^2$, the constant $\frac{s_{g_0}}{4}$ is an eigenvalue of the Laplacian of g_0 with $Ker(s_{g_0}'^*) = 0$.

On the other hand, let $\mathcal{M}_1 \subset \mathcal{M}$ the set of smooth Riemannian structures on M^n of volume 1. Given a metric $g \in \mathcal{M}_1$, let $s_g : M^n \to \mathbb{R}$ be its scalar curvature

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and dv_g the volume form determined by the metric and orientation. The total scalar curvature functional $\mathcal{S} : \mathcal{M}_1 \to \mathbb{R}$ is defined by

$$\mathcal{S}(g) = \int_{M^n} s_g dv_g$$

Due to the resolution of Yamabe problem, we may consider the set of constant scalar curvature(csc, hereafter) metrics

$$\mathcal{C} = \{g \in \mathcal{M}_1 \mid s_g \text{ constant } \}$$

(For more detail, we refer to [2],[7].) Then it has been conjectured that the set $\mathcal{C}$ of csc-metrics is rich enough so that the critical points of the total scalar curvature functional $\mathcal{S}$ restricted to $\mathcal{C}$ are also Einstein metrics (Conjecture A).

The Euler-Lagrange equations for a critical point g of this restricted variational problem may be written as the following critical point equation(CPE, hereafter):

$$z_g = s_g^*(f) \tag{1}$$

where z_g is the traceless Ricci tensor, f is a function on M^n with vanishing mean value, and $\Delta_g f = -\frac{s_g}{n-1} f$. Here, s_g is a constant. Conjecture A claims that g is Einstein, or $z_g \equiv 0$. Clearly, this implies that $f \in Ker(s_g'^*)$, or $Ker(s_g'^*) \neq 0$. Thus, if $Ker(s_g'^*) = 0$, our conjecture fails.

This paper can be considered as the second one next to [6]. It surveys the recent developments of the study of Conjecture A. For now, it is not yet clear when $Ker(s_g'^*) \neq 0$. If $Ker(s_g'^*) \neq 0$, then Conjecture A may hold.

2. Main Results

From now on, we assume that $Ker(s_g'^*) \neq 0$. Let $\varphi \in Ker(s_g'^*)$ be a non-zero function on M^n. The following Lemma is clear from [4].

Lemma 1. *The set $\Gamma = \varphi^{-1}(0)$ is a totally geodesic submanifold of M^n.*

The following lemma is essentially due to [12], c.f. [7];

Lemma 2. *If $n = 3$, at least one connected component of Γ is homeomorphic to S^2.*

Now the following theorem will show the importance of the study of the submanifold Γ in Lemma 1 and 2. First we define the Wely-Schouten tensor field on M^3 as follows; let $H = r_g - \frac{s_g}{4} g$ and define $d^D H$ to be the differential operator from $C^\infty(S^2(M))$ into $\Lambda^2 M \otimes T^* M$, given by

$$d^D H(x, y, z) = D_x H(y, z) - D_y H(x, z)$$

Theorem 2. [8] *If Weyl-Schouten tensor $d^D H$ vanished on Γ, then (M^3, g) is isometric to a $3-$dimensional standard sphere.*

Sketch of proof. In the first, we derive three relations involving $|z|^2$. The following equation hold on M^3;

$$\begin{aligned} 8W(1+f)^2|z|^2 &= (1+f)^4|d^D H|^2 + 3|dW + \frac{sf}{3} df|^2 \\ 8\Phi\varphi^2|z|^2 &= \varphi^4|d^D H|^2 + 3|d\Phi + \frac{s\varphi}{3} d\varphi|^2 \end{aligned}$$

where $W = |df|^2$, $\Phi = |d\varphi|^2$, and on Γ the above equations are reduced to

$$|z|^2 = \frac{3}{2} z(N_\varphi, N_\varphi)^2 \tag{2}$$

where $N_\varphi = \Phi^{-1/2} d\varphi$. It turns out that the equation (2) implies that z_g is diagonalized on Γ; on Γ we have

$$z(e_1, e_1) = z(e_2, e_2) = -\frac{1}{2} z(N_\varphi, N_\varphi) \quad \text{and} \quad z(e_i, e_j) = 0 \quad \text{for } i \neq j \tag{3}$$

In the second, it is shown that we can choose a solution function f_α of CPE (1) such that its gradient df_α is tangent to a connected component Γ_α of Γ in virtue of Lemma 1. Let $\Gamma_\alpha^r = \{x \in \Gamma_\alpha |\, df_\alpha \neq 0\}$. Then using this fact and (3), it may be shown that

$$6W_\alpha |z|^2 = h_\alpha^2 |d^D H|^2 \quad \text{on } \Gamma_\alpha^r \tag{4}$$

where $W_\alpha = |df_\alpha|^2$ and $h_\alpha = 1 + f_\alpha$.

Finally, applying the assumption to (3) and in virtue of the fact that $|z|^2$ is constant on $\Gamma_\alpha \setminus \Gamma_\alpha^r$, it is shown that $z_g = 0$ on Γ_α, and on M^3 as well. □

The following Theorem 3 and 4 shows the relationship between $H_2(M^3, \mathsf{Z})$ and the connectedness of the submanifold Γ.

Theorem 3. [9] *Assume that* $H_2(M^3, \mathsf{Z}) = 0$. *Then* M^3 *is diffeomorphic to* S^3 *and* Γ *is connected.*

Sketch of proof. Let $M_{0,\varphi} = \{x \in M^3 | \varphi(x) < 0\}$ and $M_{0,\varphi,i}$ be a connected component of $M_{0,\varphi}$ for $i = 1, .., k$. Also, let $M_\varphi^0 = \{x \in M^3 | \varphi(x) > 0\}$ and $M_\varphi^{0,j}$ be a connected component of M_φ^0 for $j = 1, ..., k'$. Then it can be shown that either $M_{0,\varphi,i}$ is diffeomorphic to a $3-$ball with S^2 as boundary, or there is a compact stable minimal surface Σ in $\overline{M_{0,\varphi,i}}$ in virtue of Lemma 2. The same conclusion holds for $M_\varphi^{0,j}$.

Now we consider $\check{M} = M^3 \setminus \Gamma$. Then $\check{M}$ is a disjoint union of $M_{0,\varphi}$ and M_φ^0, where $M_{0,\varphi} = \cup_{i=1}^k M_{0,\varphi,i}$, and $M_\varphi^0 = \cup_{j=1}^{k'} M_\varphi^{0,j}$. Then each of $M_{0,\varphi,i}$ and $M_\varphi^{0,j}$ is diffeomorphic to $3-$balls with S^2 as boundaries, since the existence of the compact oriented stable minimal surface Σ in each of D_i and D^j is excluded due to the fact that $H_2(M^3, \mathsf{Z}) = 0$.

In the next, noting that M^3 can be obtained by gluing $M_{0,\varphi}$ and M_φ^0 along their common boundary $\Gamma = \partial M_{0,\varphi} = \partial M_\varphi^0$, each of $M_{0,\varphi}$ and M_φ^0 is connected; that is, $k = k' = 1$ in virtue of Lemma 1.

As a consequence of the above facts, we may conclude that Γ is connected and homeomorphic to S^2. Hence, gluing $\overline{M_{0,\varphi}}$ and $\overline{M_\varphi^0}$ along $\Gamma \simeq S^2$ gives a manifold diffeomorphic to S^3. □

It turns out that the converse of Theorem 3 also holds;

Theorem 4. [9] *Assume that* Γ *is connected. Then* $H_2(M^3, \mathsf{Z}) = 0$. *In fact,* M^3 *is diffeomorphic to* S^3.

Sketch of proof. In the first, we prove that there is no embedded compact oriented stable minimal surface Σ in $\overline{M_{0,\varphi}}$ (or in $\overline{M^0_\varphi}$). Suppose that there exists an embedded compact oriented stable minimal surface Σ in $\overline{M_{0,\varphi}}$. Then we have the following results;

(i) $\Sigma = \Gamma$. It follows from the calculation that $\int_\Sigma \varphi = 0$.
(ii) The mean value of f over Σ vanishes; $\int_\Sigma f = 0$.
(iii) The stability condition of Σ gives the desired contradiction.

For the proof of (i), we have the following two equations;

$$\varphi r(\nu,\nu) = -\Delta_\Sigma \varphi$$

$$(1+f)r(\nu,\nu) = -\Delta_\Sigma f + \frac{s}{3}$$

Combining these two equations gives

$$-\int_\Sigma (1+f)\Delta_\Sigma \varphi = \int_\Sigma \varphi(1+f)r(\nu,\nu) = \int_\Sigma -\varphi\Delta_\Sigma f + \frac{s}{3}\varphi$$

Since

$$\int_\Sigma (1+f)\Delta_\Sigma \varphi = \int_\Sigma \varphi\Delta_\Sigma f$$

the equation (i) follows. The proof of (ii) are similar.

Now we prove (iii); the stability condition of Σ gives

$$\int_\Sigma (1+f)^2(r_g(\nu,\nu) + |II|^2) \le \int_\Sigma |\nabla(1+f)|_g^2 = \int_\Sigma |\nabla f|_g^2 \tag{5}$$

where $|II|^2$ is the length of the second fundamental form of Σ and $|\cdot|_g$ is a norm from the induced metric on Σ. However, we also have

$$\int_\Sigma (1+f)^2 r_g(\nu,\nu) = \int_\Sigma |\nabla f|_g^2 + \frac{s_g}{3}(1+f) \tag{6}$$

Also it follows from (i) that $\Sigma = \Gamma$ is totally geodesic, or $|II|^2 = 0$. Now, substitution of (6) into (5) gives

$$\int_\Sigma 1 + f \le 0 \tag{7}$$

Hence, in virtue of (ii) and (7), we have

$$0 < Area(\Sigma) = \int_\Sigma 1 = \int_\Sigma 1 + f \le 0$$

which is a contradiction.

Consider $\check{M} = M^3 \setminus \Gamma$, where $\check{M}$ is a disjoint union of $M_{0,\varphi}$ and M^0_φ. The contradiction above shows that there is no compact stable minimal surface in $\overline{M_{0,\varphi}}$ (or $\overline{M^0_\varphi}$), and therefore each connected components of $M_{0,\varphi}$ and M^0_φ are diffeomorphic to 3−balls with S^2 as boundaries. Gluing $M_{0,\varphi}$ and M^0_φ along their common boundary Γ, we have a manifold M^3 diffeomorphic to S^3, completing the proof of our Theorem. □

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