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CONTACT GEOMETRY WITH THE TWO ASSOCIATED STRUCTURES

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Abstract. For a contact manifold (M, η), we have a one-to-one correspondence between the Riemannian structures (η, g) and the CR-structures (η, L). It is interesting to study the interaction between the two associated structures. In this context, we approach the geometry of contact Riemannian manifolds in connection with their associated CR-structures. This paper gives a survey of recent results about this subject.

1. Introduction

A *contact manifold* (M, η) is a smooth manifold M^{2n+1} together with a global one-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. Due to Frobenius theorem we see that the contact manifold admits a maximally non-integrable hyperplane distribution D defined by the kernel of η. It is well-known that given η there exists a unique vector field ξ such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$. Moreover, there exists an associated Riemannian structure (metric) g and a (1,1)-type tensor ϕ, where g and ϕ are canonically related. We call the pair (η, g) a contact Riemannian structure and $M = (M; \eta, g)$ a contact Riemannian manifold. S. Sasaki and Y. Hatakeyama ([15], [16]) defined the normality of the contact Riemannian structure (see section 2). A normal contact Riemannian manifold is said to be a Sasakian manifold. In [12] it was proved that a Sasakian manifold which is locally symmetric ($\nabla R = 0$) must have constant curvature $+1$, where ∇ is the Levi-Civita connection. This fact means that local symmetry is a too restrictive condition for a Sasakian manifold. For this reason, T. Takahashi ([17]) introduced the notion of Sasakian locally ϕ-symmetric spaces which may be considered as the analogues of locally Hermitian symmetric spaces. He proves that this condition is equivalent to having ϕ-geodesic symmetries which are local automorphisms. It is proved in [6] that the isometry property of the ϕ-geodesic symmetry is already sufficient. For a broader class of contact Riemannian manifold we have two generalizations of the notion of locally ϕ-symmetric space. In [4] a contact Riemannian manifold is locally ϕ-symmetric if it satisfies the same curvature condition as the Sasakian case. In [7], the author give a second definition of locally ϕ-symmetric contact Riemannian manifold, that is, the characteristic reflections (namely, reflections with respect to the integral curves of ξ) are local isometries. For the first one it is called a locally ϕ-symmetric space *in the weak sense*, and for the second one a locally ϕ-symmetric space *in the strong sense* (see [7]).

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One the other hand, a contact Riemannian manifold $M = (M;\eta, g)$ has a CR-structure which is given by the holomorphic subbundle $\mathcal{H} = \{X - i\bar{\phi}X : X \in D\}$ of the complexification $TM^{\mathbb{C}}$ of the tangent bundle TM, where $\bar{\phi} = \phi|D$, the restriction of ϕ to D. Then we see that each fiber $\mathcal{H}_x$ $(x \in M)$ is of complex dimension n and $\mathcal{H} \cap \bar{\mathcal{H}} = \{0\}$, and further we have $\mathbb{C}D = \mathcal{H} \oplus \bar{\mathcal{H}}$. We say that *the associated CR-structure is integrable* if $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$. For $\mathcal{H}$ we define the *Levi form* by $L : D \times D \to \mathcal{F}(M)$, $L(X, Y) = -d\eta(X, \phi Y)$ where $\mathcal{F}(M)$ denotes the algebra of differential functions on M. Then we see that the Levi form is Hermitian and positive definite, that is, the CR-structure is *strongly pseudo-convex, pseudo-Hermitian CR-structure.* In fact, for a contact manifold $(M;\eta)$, there is a correspondence between the contact Riemannian structure (η, g) and strongly pseudo-convex, pseudo-Hermitian CR-structure (η, L) by the relation $g = L + \eta \otimes \eta$, where we denote by the same letter L the natural extension of the Levi form to a (0,2)-tensor field on M. N. Tanaka [18] defined the canonical affine connection on a nondegenerate integrable CR-manifold. In [20], S. Tanno defined the generalized Tanaka connection $\hat{\nabla}$ on a contact Riemannian manifold and further, he proved that for a given contact Riemannian manifold M the associated CR-structure is strongly pseudo-convex integrable if and only if M satisfies the integrability condition $Q = 0$ (see section 2), in which case the connection $\hat{\nabla}$ coincides with the Tanaka connection. Here, we note that the normality of a contact Riemannian structure implies the integrability of the associated CR-structure, but the converse does not always hold. The associated CR-structures of 3-dimensional contact Riemannian manifolds are always integrable (see [20]). Also, we see that their associated CR-structures are integrable for (contact Riemannian) (k, μ)-spaces (see [1] or [5]).

It is intriguing to study the geometry of a given contact Riemannian manifold $(M;\eta, g)$ in connection with the associated CR-structure, particularly with the generalized Tanaka connection. In this context, we define the Jacobi-type operator $R_{\dot\gamma} = R(\cdot, \dot\gamma)\dot\gamma$ along a unit $\hat{\nabla}$-geodesic γ. Here, we observe that the geodesics of the Levi-Civita connection and the generalized Tanaka connection do not coincide in general. In the previous works [9], [11] we have introduced and studied a new class of contact Riemannian manifolds satisfying the condition (C), i.e., the Jacobi operator field $R_{\dot\gamma}$ is diagonalizable by a $\hat{\nabla}$-parallel orthonormal frame field along γ and its eigenvalues are constant along γ, or equivalently,

$$\text{(C)} \qquad (\hat{\nabla}_{\dot\gamma} R)(\cdot, \dot\gamma)\dot\gamma = 0$$

for any unit $\hat{\nabla}$-geodesic γ, where $\hat{\nabla}$ is the generalized Tanaka connection. In [9], we have shown that $(k, 2)$-spaces $(k \neq 1)$, including the standard contact Riemannian structure of the unit tangent sphere bundle T_1M of M with constant curvature -1, are examples that are neither Sasakian nor locally symmetric but satisfy the condition (C) for any $\hat{\nabla}$-geodesic γ. Also, it is remarkable that a (k, μ)-space with $k = \mu = 0$ of dimension ≥ 5, which is a product of $(n+1)$-dimensional flat manifold and n-dimensional space of constant curvature 4, is locally symmetric but M fails to satisfy the condition (C) for any $\hat{\nabla}$-geodesic γ. In this note, we develop Theorem A in [9] by proving that a (k, μ)-space is locally ϕ-symmetric in the weak sense (Proposition 3.4). We have

Theorem A.([11]) *Let M be a (k, μ)-space. Then M satisfies the condition* (*C*) *for any $\hat{\nabla}$-geodesic γ if and only if* (1) $k = 1$ *and M is Sasakian locally ϕ-symmetric or* (2) $\mu = 0$ *in which case M is 3-dimensional, or* (3) $\mu = 2$.

In [9], we also proved that the standard contact Riemannian structure of the unit tangent sphere bundle T_1M of a 2-dimensional Riemannian manifold M satisfies condition (C) for any $\hat{\nabla}$-geodesic γ if and only if M has constant Gauss curvature 1, 0 or -1. We proved this result for an arbitrary dimension. Namely, we proved in [11]

Theorem B. *Let M be a $(n+1)$-dimensional Riemannian manifold. Then the standard contact Riemannian structure of the unit tangent sphere bundle T_1M satisfies the condition (C) for any $\hat{\nabla}$-geodesic γ if and only if the base manifold M is of constant curvature $c = 1$, $n = 1$ and $c = 0$, or $c = -1$.*

Remark 1. It was proved in [3] ([13] and [14], respectively) that the base manifold is of constant curvature $c = -1$, $c = 1$ ($c = 0$, $c = 1$, respectively) if and only if the standard contact Riemannian structure on the unit tangent sphere bundle is a critical point of some functional on the set of associated Riemannian metrics $\mathcal{M}(\eta)$ of a given contact form η.

Moreover, we give a complete classification of three-dimensional contact Riemannian manifolds satisfying the condition (C) for any $\hat{\nabla}$-geodesic γ. More precisely, we prove ([11])

Theorem C (local classification). *Let M be a 3-dimensional contact Riemannian manifold. Then M satisfies the condition (C) for any $\hat{\nabla}$-geodesic γ if and only if M is locally isometric to one of the following spaces:*

(1) *a Sasakian ϕ-symmetric space;*

(2) $SU(2)$ *(or $SO(3)$), $SL(2,\mathbb{R})$ (or $O(1,2)$) with a special left-invariant contact metric which is not Sasakian, respectively;*

(3) *a flat manifold.*

In [6] the authors gave a classification of Sasakian ϕ-symmetric spaces (complete and simply connected Sasakian locally ϕ-symmetric spaces). Together with this classification we have ([11])

Theorem D (global classification). *Let M be a complete and simply connected 3-dimensional contact Riemannian manifold. Then M satisfies the condition (C) for any $\hat{\nabla}$-geodesic γ if and only if M is isometric to one of the following spaces:*

(1) *the standard unit sphere S^3; $SU(2)$, $\widetilde{SL(2,\mathbb{R})}$ (the universal covering of $SL(2,\mathbb{R})$) or the Heisenberg group H with a left-invariant Sasakian metric, respectively;*

(2) *$SU(2)$, $\widetilde{SL(2,\mathbb{R})}$ with a special left-invariant contact metric which is not Sasakian, respectively;*

(3) $\mathbb{R}^3$.

For the proofs of above Theorem B, Theorem C and Theorem D, we refer to [11].

2. Preliminaries

First, we review some fundamental material about contact Riemannian geometry and refer to [1], [2] for further detailed treatments. All manifolds in the present note are assumed to be connected and smooth.

A $(2n+1)$-dimensional manifold M^{2n+1} is said to be a contact manifold if it admits a global one-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form η, there exists a unique vector field ξ, called the characteristic vector field, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X. It is well-known that there also exists a Riemannian metric g and a $(1,1)$-tensor field ϕ such that

$$\phi^2 X = -X + \eta(X)\xi, \ d\eta(X,Y) = g(X, \phi Y), \ g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.1}$$

where X and Y are vector fields on M. From (2.1), it follows that

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi). \tag{2.2}$$

A Riemannian manifold M equipped with structure tensors (η, g) satisfying (2.1) is said to be a *contact Riemannian manifold* or *contact metric manifold* and it is denoted by $M = (M; \eta, g)$. Given a contact Riemannian manifold M, we define a $(1,1)$-tensor field h by $h = \frac{1}{2} L_\xi \phi$, where L denotes Lie differentiation. Then we may observe that h is symmetric and satisfies

$$h\xi = 0, \quad h\phi = -\phi h, \tag{2.3}$$

$$\nabla_X \xi = -\phi X - \phi h X, \tag{2.4}$$

where ∇ is Levi-Civita connection. From (2.3) and (2.4), we see that each trajectory of ξ is a geodesic. A contact Riemannian manifold for which ξ is a Killing vector field, is called a K-contact manifold. It is easy to see that a contact Riemannian manifold is K-contact if and only if $h = 0$. For a contact Riemannian manifold M, one may define naturally an almost complex structure J on $M \times \mathbb{R}$ by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}),$$

where X is a vector field tangent to M, t the coordinate of $\mathbb{R}$ and f a function on $M \times \mathbb{R}$. If the almost complex structure J is integrable, M is said to be normal or Sasakian. It is known that M is normal if and only if M satisfies

$$[\phi, \phi] + 2d\eta \otimes \xi = 0,$$

where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ. A Sasakian manifold is also characterized by the condition

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X$$

for all vector fields X and Y on the manifold and this is equivalent to

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y$$

for all vector fields X and Y (cf. [1],[2]). Here, R is the Riemannian curvature tensor defined by

$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$$

for all vector fields X, Y, Z.

Now, we review the *generalized Tanaka connection* ([20]) on a contact Riemannian manifold $M = (M; \eta, g)$. The generalized Tanaka connection $\hat{\nabla}$ is defined by

$$\hat{\nabla}_X Y = \nabla_X Y + \eta(X)\phi Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi$$

for all vector fields X and Y on M. Together with (2.4), $\hat{\nabla}$ may be rewritten as

$$\hat{\nabla}_X Y = \nabla_X Y + \eta(X)\phi Y + \eta(Y)(\phi X + \phi h X) - g(\phi X + \phi h X, Y)\xi \tag{2.5}$$

and we see that the generalized Tanaka connection $\hat{\nabla}$ has the torsion $\hat{T}(X,Y) = 2g(X, \phi Y)\xi + \eta(Y)\phi h X - \eta(X)\phi h Y$. We put

$$A(X,Y) = \eta(X)\phi Y + \eta(Y)(\phi X + \phi h X) - g(\phi X + \phi h X, Y)\xi$$

for all vector fields X and Y on M. Then A is a (1,2)-tensor field on M and $\hat{\nabla}_X Y = \nabla_X Y + A(X,Y)$. In particular, for a K-contact Riemannian manifold we get

$$A(X,Y) = \eta(X)\phi Y + \eta(Y)\phi X - g(\phi X, Y)\xi,$$

where X and Y are vector fields. For a given contact Riemannian manifold M the associated CR-structure is strongly pseudo-convex integrable if and only if M satisfies the integrability condition $Q = 0$, where Q is a (1,2)-tensor field on M defined by

$$Q(X,Y) = (\nabla_X \phi)Y - g(X + hX, Y)\xi + \eta(Y)(X + hX) \tag{2.6}$$

for all vector fields $X,\ Y$ on M (see [20, Proposition 2.1]). Further, it was proved that

Proposition 2.1 ([20]). *The generalized Tanaka connection $\hat{\nabla}$ on a contact Riemannian manifold $M = (M; \eta, g)$ is the unique linear connection satisfying the following conditions:*

(i) $\hat{\nabla}\eta = 0$, $\hat{\nabla}\xi = 0$;
(ii) $\hat{\nabla} g = 0$;
(iii − 1) $\hat{T}(X,Y) = 2d\eta(X,Y)\xi$, $X,\ Y \in D$;
(iii − 2) $\hat{T}(\xi, \phi Y) = -\phi \hat{T}(\xi, Y)$, $Y \in D$;
(iv) $(\hat{\nabla}_X \phi)Y = Q(X,Y)$, $X,\ Y \in TM$.

The Tanaka connection ([18]) on a nondegenerate integrable CR-manifold is defined as the unique linear connection satisfying (i), (ii), (iii-1), (iii-2) and $\hat{\nabla}\phi = 0$. The metric affine connection $\hat{\nabla}$ is a naturally generalization of the Tanaka connection.

Let γ be a $\hat{\nabla}$-geodesic parametrized by the arc-length parameter s, where a $\hat{\nabla}$-geodesic means a geodesic with respect to $\hat{\nabla}$. From (2.5) we see that a $\hat{\nabla}$-geodesic does not coincide with a ∇-geodesic. Define the Jacobi operator $R_{\dot\gamma}$ by $R_{\dot\gamma} = R(\cdot, \dot\gamma)\dot\gamma$ along γ, where $\dot\gamma$ is the unit tangent vector field of γ. Then $R_{\dot\gamma}$ is a symmetric $(1,1)$-tensor field along γ. Moreover, from (i) of Proposition 2.1 we observe that $\eta(\dot\gamma)$ is constant along γ, and thus a $\hat{\nabla}$-geodesic whose tangent initially belongs to D remains in D. We call such a $\hat{\nabla}$-geodesic which is tangent to D a *horizontal* $\hat{\nabla}$*-geodesic.*

We recall the definition of a Sasakian locally ϕ-symmetric space ([17]).

Definition 2.2. A Sasakian manifold $M = (M; \eta, g)$ is said to be locally ϕ-symmetric if $\phi^2(\nabla_V R)(X,Y)Z = 0$ for all vector fields $V, X, Y, Z \in D$.

As a generalization of the above Sasakian one, a contact Riemannian locally ϕ-symmetric space is defined in [4] by the same condition and is called ([7]) a *locally ϕ-symmetric space in the weak sense*. We have proved in [9]:

Theorem 2.3. *A Sasakian manifold M is locally ϕ-symmetric if and only if M satisfies the condition (C) for any horizontal $\hat{\nabla}$-geodesic γ, or if and only if M satisfies the condition (C) for any $\hat{\nabla}$-geodesic γ.*

3. A contact (k,μ)-space

In [5], the (k,μ)-nullity distribution of a contact Riemannian manifold M, for the pair $(k,\mu) \in \mathbb{R}^2$, is defined by

$$N(k,\mu) : p \to N_p(k,\mu) = \{z \in T_pM | R(x,y)z = k(g(y,z)x - g(x,z)y) + \mu(g(y,z)hx - g(x,z)hy) \text{ for any } x, y \in T_pM\}.$$

A *(k,μ)-space* is a contact Riemannian manifold with ξ belonging to the (k,μ)-nullity distribution, that is,

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{3.1}$$

and from the symmetry of the curvature tensor, we obtain

$$R(\xi,X)Y = k(g(X,Y)\xi - \eta(Y)X) + \mu(g(hX,Y)\xi - \eta(Y)hX). \tag{3.2}$$

It was shown in [5] that the (k,μ)-spaces are invariant under a D-homothetic deformation. As mentioned previously, the associated CR-structures of the (k,μ)-spaces are integrable, that is, $Q = 0$. This class contains Sasakian manifolds ($k = 1$ and $h = 0$). The unit tangent sphere bundle is a (k,μ)-space if and only if the base manifold is of constant curvature c with $k = c(2-c)$ and $\mu = -2c$ ([5]). (By virtue of the result of Y. Tashiro [21], we know that for $c \neq 1$, the unit tangent sphere bundle is non-Sasakian.) Very recently, E. Boeckx [8] presented explicit examples for all possible dimensions and all possible (k,μ). Furthermore, in [5] they proved the following result.

Theorem 3.1. Let $M = (M; \eta, g)$ be a contact Riemannian manifold. If ξ belong to the (k,μ)-nullity distribution, then $k \leq 1$. If $k = 1$, then $h = 0$ and M is a Sasakian manifold. If $k < 1$, then M admits three mutually orthogonal and integrable distributions $D(0)$, $D(\lambda)$ and $D(-\lambda)$, defined by the eigenspaces of h, where $\lambda = \sqrt{1-k}$. Moreover

$$\begin{aligned}
&R(X_\lambda, Y_\lambda)Z_{-\lambda} = (k-\mu)\{g(\phi Y_\lambda, Z_{-\lambda})\phi X_\lambda - g(\phi X_\lambda, Z_{-\lambda})\phi Y_\lambda\},\\
&R(X_{-\lambda}, Y_{-\lambda})Z_{\lambda} = (k-\mu)\{g(\phi Y_{-\lambda}, Z_{\lambda})\phi X_{-\lambda} - g(\phi X_{-\lambda}, Z_{\lambda})\phi Y_{-\lambda}\},\\
&R(X_\lambda, Y_{-\lambda})Z_{-\lambda} = kg(\phi X_\lambda, Z_{-\lambda})\phi Y_{-\lambda} + \mu g(\phi X_\lambda, Y_{-\lambda})\phi Z_{-\lambda},\\
&R(X_\lambda, Y_{-\lambda})Z_{\lambda} = -kg(\phi Y_{-\lambda}, Z_{\lambda})\phi X_\lambda - \mu g(\phi Y_{-\lambda}, X_\lambda)\phi Z_\lambda,\\
&R(X_\lambda, Y_\lambda)Z_\lambda = \{2(1+\lambda) - \mu\}\{g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda\},\\
&R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = \{2(1-\lambda) - \mu\}\{g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}\},
\end{aligned}$$

where X_λ, Y_λ, $Z_\lambda \in D(\lambda)$ and $X_{-\lambda}$, $Y_{-\lambda}$, $Z_{-\lambda} \in D(-\lambda)$.

In proving Theorem 3.1, the following proposition has shown to be useful:

Proposition 3.2 Suppose $k < 1$. Then we have :
(i) If $X, Y \in D(\lambda)$ (resp.$\in D(-\lambda)$), then $\nabla_X Y \in D(\lambda)$ (resp.$D(-\lambda)$)
(ii) If $X \in D(\lambda)$, $Y \in D(-\lambda)$, then $\nabla_X Y$(resp. $\nabla_Y X$) has no component in $D(\lambda)$ (resp. $D(-\lambda)$).

We prove in [10] that a locally symmetric contact (k, μ)-space is locally the product space of flat $(n+1)$-dimensional manifold and n-dimensional manifold of positive constant curvature 4, or a space of constant curvature 1, and in which case the structure is Sasakian.

Furthermore, we have proved in [9]:

Theorem 3.3. *Let M be a (k, μ)-space. If M satisfies the condition (C) for any $\hat{\nabla}$-geodesic γ, then we have:*
(i) $k = 1$ and M is a Sasakian locally ϕ-symmetric space;
(ii) $\mu = 0$ and M is a 3-dimensional locally ϕ-symmetric space in the weak sense;
(iii) $\mu = 2$ and M is a locally ϕ-symmetric space in the weak sense.

In [7] it has been proved that all (k, μ)-spaces are locally ϕ-symmetric in the strong sense, i.e., the characteristic reflections are local isometries, and hence also in the weak sense. Thus, we have

Proposition 3.4. *A (k, μ)-space $(k < 1)$ is locally ϕ-symmetric in the weak sense.*

Here, we provide an alternative proof of this property.

Proof. At first, we see that from (2.1) and Definition 2.2 M is locally ϕ-symmetric if and only if $g((\nabla_V R)(X, Y)Z, W) = 0$ for all $V, X, Y, Z, W \in D$. By using Propositions 3.1, 3.2, (2.4), (2.6) $(Q = 0)$ and by straightforward calculations we have following equations:

$$\begin{aligned}(\nabla_{V_\lambda} R)(X_\lambda, Y_\lambda)Z_{-\lambda} =& - g(Z_{-\lambda}, \phi V_\lambda + \phi h V_\lambda)R(X_\lambda, Y_\lambda)\xi \\ &+ (k-\mu)\{g(\phi Y_\lambda, Z_{-\lambda})g(V_\lambda + hV_\lambda, X_\lambda)\xi \\ &- g(\phi X_\lambda, Z_{-\lambda})g(V_\lambda + hV_\lambda, Y_\lambda)\xi\},\end{aligned}$$

$$\begin{aligned}(\nabla_{V_{-\lambda}} R)(X_\lambda, Y_\lambda)Z_{-\lambda} =& - g(X_\lambda, \phi V_{-\lambda} + \phi h V_{-\lambda})R(\xi, Y_\lambda)Z_{-\lambda} \\ &- g(Y_\lambda, \phi V_{-\lambda} + \phi h V_{-\lambda})R(X_\lambda, \xi)Z_{-\lambda} \\ &+ (k-\mu)\{g(\phi Y_\lambda, Z_{-\lambda})g(V_{-\lambda} + hV_{-\lambda}, X_\lambda)\xi \\ &- g(\phi X_\lambda, Z_{-\lambda})g(V_{-\lambda} + hV_{-\lambda}, Y_\lambda)\xi\},\end{aligned}$$

$$\begin{aligned}(\nabla_{V_\lambda} R)(X_{-\lambda}, Y_{-\lambda})Z_\lambda =& - g(X_{-\lambda}, \phi V_\lambda + \phi h V_\lambda)R(\xi, Y_{-\lambda})Z_\lambda \\ &- g(Y_{-\lambda}, \phi V_\lambda + \phi h V_\lambda)R(X_{-\lambda}, \xi)Z_\lambda \\ &+ (k-\mu)\{g(\phi Y_{-\lambda}, Z_\lambda)g(V_\lambda + hV_\lambda, X_{-\lambda})\xi \\ &- g(\phi X_{-\lambda}, Z_\lambda)g(V_\lambda + hV_\lambda, Y_{-\lambda})\xi\},\end{aligned}$$

$$\begin{aligned}(\nabla_{V_{-\lambda}} R)(X_{-\lambda}, Y_{-\lambda})Z_\lambda =& - g(Z_\lambda, \phi V_{-\lambda} + \phi h V_{-\lambda})R(X_{-\lambda}, Y_{-\lambda})\xi \\ &+ (k-\mu)\{g(\phi Y_{-\lambda}, Z_\lambda)g(V_{-\lambda} + hV_{-\lambda}, X_{-\lambda})\xi \\ &- g(\phi X_{-\lambda}, Z_\lambda)g(V_{-\lambda} + hV_{-\lambda}, Y_{-\lambda})\xi\},\end{aligned}$$

$$\begin{aligned}(\nabla_{V_\lambda}R)(X_\lambda,Y_{-\lambda})Z_{-\lambda} =& -g(Y_{-\lambda},\phi V_\lambda+\phi hV_\lambda)R(X_\lambda,\xi)Z_\lambda\\ &-g(Z_{-\lambda},\phi V_\lambda+\phi hV_\lambda)R(X_\lambda,Y_{-\lambda})\xi\\ &+kg(\phi Y_{-\lambda},Z_\lambda)g(V_\lambda+hV_\lambda,X_\lambda)\xi\\ &+\mu g(\phi X_\lambda,Y_{-\lambda})g(V_\lambda+hV_\lambda,Z_{-\lambda})\xi,\end{aligned}$$

$$\begin{aligned}(\nabla_{V_{-\lambda}}R)(X_\lambda,Y_{-\lambda})Z_{-\lambda} =& -g(X_\lambda,\phi V_{-\lambda}+\phi hV_{-\lambda})R(\xi,Y_{-\lambda})Z_{-\lambda}\\ &+kg(\phi X_\lambda,Z_{-\lambda})g(V_{-\lambda}+hV_{-\lambda},Y_{-\lambda})\xi\\ &+\mu g(\phi X_\lambda,Y_{-\lambda})g(V_{-\lambda}+hV_{-\lambda},Z_{-\lambda})\xi,\end{aligned}$$

$$\begin{aligned}(\nabla_{V_\lambda}R)(X_\lambda,Y_{-\lambda})Z_\lambda =& -g(Y_{-\lambda},\phi V_\lambda+\phi hV_\lambda)R(X_{-\lambda},\xi)Z_\lambda\\ &-kg(\phi Y_{-\lambda},Z_\lambda)g(V_\lambda+hV_\lambda,X_\lambda)\xi\\ &-\mu g(\phi Y_{-\lambda},X_\lambda)g(V_\lambda+hV_\lambda,Z_\lambda)\xi,\end{aligned}$$

$$\begin{aligned}(\nabla_{V_{-\lambda}}R)(X_\lambda,Y_{-\lambda})Z_\lambda =& -g(X_\lambda,\phi V_{-\lambda}+\phi hV_{-\lambda})R(\xi,Y_{-\lambda})Z_\lambda\\ &-g(Z_\lambda,\phi V_{-\lambda}+\phi hV_{-\lambda})R(X_\lambda,Y_{-\lambda})\xi\\ &-kg(\phi Y_{-\lambda},Z_\lambda)g(V_{-\lambda}+hV_{-\lambda},X_\lambda)\xi\\ &-\mu g(\phi Y_{-\lambda},X_\lambda)g(V_{-\lambda}+hV_{-\lambda},Z_\lambda)\xi,\end{aligned}$$

$$\begin{aligned}(\nabla_{V_{-\lambda}}R)(X_\lambda,Y_\lambda)Z_\lambda =& -g(X_\lambda,\phi V_{-\lambda}+\phi hV_{-\lambda})R(\xi,Y_\lambda)Z_\lambda\\ &-g(Y_\lambda,\phi V_{-\lambda}+\phi hV_{-\lambda})R(X_\lambda,\xi)Z_\lambda\\ &+[2(1+\lambda)-\mu]\{g(\phi Y_\lambda,Z_\lambda)g(\phi V_{-\lambda}+\phi hV_{-\lambda},X_\lambda)\xi\\ &-g(\phi X_\lambda,Z_\lambda)g(\phi V_{-\lambda}+\phi hV_{-\lambda},Y_\lambda)\xi\},\end{aligned}$$

$$\begin{aligned}(\nabla_{V_\lambda}R)(X_{-\lambda},Y_{-\lambda})Z_{-\lambda} =& -g(X_{-\lambda},\phi V_\lambda+\phi hV_\lambda)R(\xi,Y_{-\lambda})Z_{-\lambda}\\ &-g(Y_{-\lambda},\phi V_\lambda+\phi hV_\lambda)R(X_{-\lambda},\xi)Z_{-\lambda}\\ &+[2(1+\lambda)-\mu]\{g(\phi Y_{-\lambda},Z_{-\lambda})g(\phi V_\lambda+\phi hV_\lambda,X_\lambda)\xi\\ &-g(\phi X_{-\lambda},Z_{-\lambda})g(\phi V_\lambda+\phi hV_\lambda,Y_{-\lambda})\xi\},\end{aligned}$$

$$(\nabla_{V_\lambda}R)(X_\lambda,Y_\lambda)Z_\lambda=(\nabla_{V_{-\lambda}}R)(X_{-\lambda},Y_{-\lambda})Z_{-\lambda}=0.$$

From the above equations, by using (3.1) and (3.2), we may conclude that a contact Riemannian manifold with ξ belonging to the (k,μ)-nullity distribution ($k<1$) is locally ϕ-symmetric.

Therefore, together with Proposition 3.4 and the proof of Theorem 3.3 (see [11]), we have Theorem A.

Remark 2. An example of a contact flat Riemannian structure on $\mathbb{R}^3(x^1,x^2,x^3)$ is given by $\eta=\frac{1}{2}(\cos x^3dx^1+\sin x^3dx^2)$ and $g_{ij}=\frac{1}{4}\delta_{ij}$. For dimension ≥ 5 a contact manifold cannot admit a contact Riemannian structure of vanishing curvature (cf. [1]). Also, it was proved that a contact Riemannian manifold M^{2n+1} which satisfy $R(X,Y)\xi=0$ for all vector fields X and Y (i.e., ξ belonging to the (0,0)-nullity distribution) is locally a product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive constant sectional curvature equal to 4. Hence, we see that a contact Riemannian manifold M^{2n+1} ($n\geq 2$) satisfying $R(X,Y)\xi=0$ is locally symmetric but it does not satisfy the condition (C) for any $\hat{\nabla}$-geodesic.

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